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Utilizing Incommensurate Sampling Rates to Layer Audio Files for Use with Variable Bandwidth

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UTILIZING INCOMMENSURATE SAMPLING RATES TO LAYER AUDIO FILES FOR USE WITH VARIABLE BANDWIDTH

By

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To Lee and Marjean

for teaching me

to live is to learn

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Abstract

In the study of Sampling Theory, the Nyquist-Shannon sampling theorem, proved in the first half of the 20th century, showed that a bandwidth limited signal could be exactly reconstructed when sampled at an appropriate rate. This is the basis for digital representation of audio files (CDs), since human hearing is bandwidth limited. Dr. Stephen Casey proved in his recent research that such a signal, sampled at multiple specifically chosen incommensurate rates, can also be exactly reconstructed. This research takes his theorem and applies it to a regularly sampled audio signal to construct new data sets, sampled at incommensurate rates. The signal can be reconstructed from one or more of these sets, with quality increasing as more data sets are included. Each data set can buffer individually when streamed over a broadband connection. Thus, when bandwidth changes the audio player can add or drop a data set without having to completely re-buffer.

Acknowledgements

First and foremost, I would like to acknowledge my thesis advisor, mentor, and professor, Dr. Stephen Casey. My research presented in this thesis is an application of his previous research; content dealing with his theorems, the introductions, and many of the sources, are derived from class notes and his papers. Without his research and his continuous support on the project, this thesis would never have come to fruition. Beyond this, a great deal of typesetting and time was saved due to my access to the TeX version of these documents.

Throughout my education at American University, Dr. Casey and Dr. Ali Enayat fundamentally shaped my mathematical maturity. My first introduction to a 'proof' was in Advanced Calculus with Dr. Casey, and it was at this moment that I decided to become a mathematician. Dr. Enayat taught my first Analysis class, introduced me to formal mathematical logic, and taught the final class of my degree: Lebesgue Measure Theory. It was in these classes that I not only developed the skills to sort out difficult concepts, but also the ability to present their proofs effectively. Dr. Enayat also helped formulate some expanded ideas on the specific classes of numbers used.

My understanding of rigor, and its importance in mathematics, is due significantly to Dr. I-Lok Chang. The size of classes I took with Dr. Chang reached a maximum in the first class at six students; I am forever gratefully for the enormous amount of personal time he has invested in me by means of independent studies and research projects. Under his instruction, I explored deep ideas derived from a small, well-stated set of axioms. Dr. Chang's influence naturally guided this thesis, especially in the presentations of the given proofs. I would also like to thank him for providing guidance and material on the ideas of generalized functions.

Dr. Joshua Lansky and Dr. Jeffrey Adler have also dedicated large portions of time to my studies, and a few other students', with independent studies and informal seminars. Exploring these new topics outside of typical course work challenged my ability to grasp new concepts and taught me new skills applicable to all mathematical analysis.

Dr. Adler also provided a comprehensive set of specific revisions, ranging from AMS hyphenation standards ("bandwidth-limited functions are bandwidth limited") to mathematical constancy throughout the thesis. A friend, Morgan Haycock, also provided a number of revisions on early drafts and read through these acknowledgements before publicaltion.

All of the staff at the College of Arts and Sciences have been critical in the technical process of formatting and submitting the thesis. They also did an excellent job organizing and executing the *Robyn Rafferty Mathias Student Research Conference* where this research was presented. It's a wonderful opportunity for motivated students to share their ideas in formal setting. Finally, I would like to thank my committee, and Department of Mathematics and Statistics at American University, for their time and contributions to this thesis: it would not be the same without them.

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1 Introduction

The origin of this research stems back to a conversation with my advisor, Dr. Stephen Casey, about a theorem he had recently proved that appeared in *The Journal of Applied Functional Analysis.* While much of the analysis in the paper was, and some still remains, beyond my mathematical skill set, I immediately thought of an application to digital audio signals. Originally, I had anticipated an ability to compress audio further than current encoding schemes have achieved. In fact, in March 2009 at the *19th Annual Robyn Rafferty Mathias Student Research Conference*, I presented preliminary research titled *Audio Compression Using Incommensurate Sampling Rates.* However, as my research continued, the potential to compress audio using this type of analysis declined. The amount of compression that could be achieved seemed to be minimal and required a relational data structure that was far beyond my computer science skills.

The process would have taken one audio file and convert it into a few audio files which, together, would be roughly the same size as the original. Compression would have come from eliminating near overlap between these new files. When this became an impractical project, I proposed using this multiple file structure as a type of layering of an audio signal. This is useful in a variable bandwidth setting, such as streaming audio to a computer or a cell phone. This paper reviews introductory sampling theory, states Casey's multichannel deconvolution theorem, describes current digital to audio conversion algorithms, and produces an original algorithm for creating and reconstructing multichannel layered audio files.

1.1 Background

Casey's research [Cas07] undertook the problem of determining the flatness of a surface, such as a silicon chip. Computer chip manufacturers require a piece of silicon to be sufficiently smooth in order to add grooves in which the electrons flow. The smoother the silicon surface, the closer these grooves can be placed; a shorter distance for the electrons to travel produces a faster chip. The process to test the smoothness uses a small needle, which is placed above the surface. An electrical charge is run through a needle and its reflection off the surface is measured. A smaller needle allows for a more precise measurement and, due to the economic and technological advances sustaining Moore's Law[1], smaller and smaller needles have been used. The needles, however, have a lower bound on size with respect to the ability to carry an electrical charge. Engineers reached a point where any smaller needles would burst whenever the necessary electrical charge was applied.

The solution to this problem came in the form of complex and harmonic analysis. The problem, in essence, is the reconstruction of a function (in this case we have the function of the distance between the needle and the surface) from discrete data values. It has been well known that this is possible for bandwidth limited functions when sampling above the Nyquist rate. In this setting, the 'rate' does not refer to time, but rather the size of the needle. Casey showed that by measuring with multiple needles with relative radii proportional to the integers[2], the data could reconstruct the function to a level equivalent to measuring with a needle smaller than each of these multiple needles.

This discussion immediately led me to question: could this process also be used with audio signals? The answer was a quick and obvious yes, and his paper had actually dealt with this situation. I began to consider the possible implications of this concept on audio compression. As mentioned above, the first idea dealt with compression which did not pan out. Instead, the idea to use this theorem to improve streaming audio technology was conceived.

1.2 Preliminary Definitions

We first introduce useful definitions which shall be referenced in the paper. These are referenced in a number of texts books in complex, harmonic, and numerical analysis.

[1] Moore's Law states that computer processors double in speed roughly every 16 months. This phenomena has nearly held true since the advent of digital computation.

[2] The use of the integers here has to do with the circular nature of the needle and the zeros of Bessel functions. This paper will deal with a different set of sampling rates.

However much of this section is cited from a collection of notes arranged by my advisor, Casey. [Cas10]

Definition 1.1 *A function f is called absolutely integrable, i.e. $f \in L^1(\mathbb{R})$, if*

$$\|f\|_1 := \int_{\mathbb{R}} |f(x)|dx < +\infty.$$

If f is in L^1, we say that $\|f\|_1$ is the L^1 norm of f. Similarly, a function is called square integrable, i.e. $f \in L^2(\mathbb{R})$, if

$$\|f\|_2 := \int_{\mathbb{R}} |f(x)|^2 dx < +\infty.$$

If f is in L^2, we say that f is of finite energy and $\|f\|_2^2$ is the L^2 norm of f.

For the remainder of the paper, all functions will be considered absolutely and square integrable functions on the real line, and all integrals will be done with respect to the Lebesgue measure, unless otherwise noted. The Fourier series and Fourier transform are fundamental in the study of sampling theory and their definitions follow.

Definition 1.2 (Fourier Series) *Let f be a periodic, continuous, integrable function on $\mathbb{R}$, with period 2Ω. Then the Fourier series of f is defined by*

$$f(x) := \sum_{n \in \mathbb{Z}} c_n \exp^{-i\pi n x/\Omega}$$

where the Fourier coefficients $\{c_n\}$ are given by

$$c_n := \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} f(x) \exp^{i\pi n x/\Omega} dx.$$

Definition 1.3 (Fourier Transform and Inversion Formula) *The Fourier transform of $f \in L^1(\mathbb{R})$ is defined as*

$$\widehat{f}(\omega) := \int_{\mathbb{R}} f(t) \exp^{-2\pi i t \omega} dt$$

and its inversion formula[3], *for* $g \in L^1(\widehat{\mathbb{R}})$, *is*

$$\check{g}(t) := \int_{\mathbb{R}} g(\omega) \exp^{2\pi i \omega t} d\omega.$$

If we consider the function f to be a signal, then the domain of f is time. The domain of $\widehat{f}$, the Fourier transform of f, represents the signal frequency. The following definitions are useful when referring to the time or frequency domain of such a function and its transform:

Definition 1.4 (Support) *The support of a function* $f : \mathbb{R} \longrightarrow \mathbb{R}$, *denoted supp(f), is the closure of the set on which* f *is non-zero, i.e. supp(f)* $= \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. *The function* f *is said to have compact support if supp(f) is a compact set.*

The Fourier Transform has a number of mathematically convenient properties: linearity, symmetry, conjugation, boundedness, and continuity, to name a few. Convolution is also a relevant property to the study of sampling theory.

Definition 1.5 (Convolution) *For* $f, g \in L^1(\mathbb{R})$, *the convolution of* f *and* g *is defined by*

$$(f * g)(t) := \int_{\mathbb{R}} f(\tau) g(t - \tau) d\tau$$

for $t, \tau \in \mathbb{R}$.

Convolution on either side of the Fourier Transform is equivalent to multiplication on the other side,

$$\widehat{(f * g)} = \widehat{f} \cdot \widehat{g}, \tag{1}$$

$$\widehat{(f \cdot g)} = \widehat{f} * \widehat{g}. \tag{2}$$

[3] $\widehat{\mathbb{R}}$ is equivalent to $\mathbb{R}$ and is used to differentiate the time domain (pre-transform) from the frequency domain (post-transform).

Proof of Equation 1: Let $f, g \in L^1(\mathbb{R})$, then

$$\begin{aligned}
\widehat{(f*g)}(\omega) &= \int_{\mathbb{R}} (f*g)(t) \exp^{-2\pi i t\omega}\, dt \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(t-\tau) g(\tau)\, d\tau \right] \exp^{-2\pi i t\omega}\, dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(t-\tau) g(\tau) \exp^{-2\pi i t\omega}\, d\tau\, dt.
\end{aligned}$$

The interchange of integrals is justified by Fubini and Tonelli [Ben74]. This last equality is justified by the fact that $\exp^{-2\pi i t\omega}$ does not depend on the integrand with respect to τ. We now make a u-substitution. Let $u = t - \tau$, then

$$\left[\begin{array}{lcl} t &=& u + \tau \\ \tau &=& t - u \\ du &=& dt. \end{array} \right.$$

Note that as t ranges in $\mathbb{R}$, u ranges $\mathbb{R}$ as well, and thus the limits of integration do not change with respect to the substitution.

$$\begin{aligned}
\widehat{(f*g)}(\omega) &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) g(\tau) \exp^{-2\pi i (u+\tau)\omega}\, d\tau\, du \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) \exp^{-2\pi i u\omega} g(\tau) \exp^{-2\pi i \tau\omega}\, d\tau\, dt \\
&= \int_{\mathbb{R}} f(u) \exp^{-2\pi i u\omega} \left[\int_{\mathbb{R}} g(\tau) \exp^{-2\pi i \tau\omega}\, d\tau \right] dt \\
&= \int_{\mathbb{R}} f(u) \exp^{-2\pi i u\omega} \widehat{g}(\omega)\, du \\
&= \int_{\mathbb{R}} f(u) \exp^{-2\pi i u\omega}\, du \cdot \widehat{g}(\omega) \\
&= \widehat{f}(\omega) \cdot \widehat{g}(\omega).
\end{aligned}$$

$\square$

Equation (2) justified similarly, replacing $\exp^{-2\pi i t\omega}$ with $\exp^{2\pi i t\omega}$.

1.3 The Dirac Delta Functional

Suppose we wished to find the identity under convolution, i.e. a function g such that

$$f * g = f.$$

However, no such analytic function exists. We wish to construct a generalized function, with the following conditions:

i. $\int_{\mathbb{R}} \delta(t)\, dt = 1$

ii. $\operatorname{supp}\{\delta\} = \{0\}$

iii. $\delta(t) \geq 0$ for all $t \in \mathbb{R}$

Since the Lebesgue measure of a singleton is 0, i.e. $\lambda(\{0\}) = 0$, the first two conditions contradict each other. We can, however, create a sequence of functions and observe their action on a continuous function g in the limit. In this sense, we are constructing a generalized function. It is important to note here that convergence is in the sense of generalized functions [Lig62]. Let $\{f_n(t) : n \in \mathbb{N}\}$ be a sequence of real-valued functions such that

$$f_n(t) = \frac{n}{2}\chi_{[-\frac{1}{n},\frac{1}{n}]}(t)$$

where

$$\chi_E = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E. \end{cases}$$

Let g be a continuous function on $\mathbb{R}$. We aim to see that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(t) g(t)\, dt = g(0). \tag{3}$$

Proof of Equation (3): Let $\epsilon > 0$ be given. Since g is continuous, there exists[4] a $\gamma > 0$ such that $|g(t) - g(0)| < \epsilon$ whenever $|t - 0| < \gamma$. Let $N_\gamma = \left\lceil \frac{1}{\gamma} \right\rceil$, where $\lceil\cdot\rceil$ denotes the

[4] γ used here instead of the traditional δ as not to confuse any part of the proof with the δ functional.

ceiling function. Then for $n > N_\gamma$, $n > \frac{1}{\gamma}$ and

$$\begin{aligned}
\left|\int_{\mathbb{R}} g(t) f_n(t)\, dt - g(0)\right| &= \left|\int_{\mathbb{R}} g(t) \frac{n}{2} \chi_{[-\frac{1}{n},\frac{1}{n}]}(t)\, dt - g(0)\right| \\
&= \left|\frac{n}{2}\int_{-1/n}^{1/n} g(t)\, dt - \frac{n}{2}\frac{2}{n} g(0)\right| \\
&= \left|\frac{n}{2}\int_{-1/n}^{1/n} g(t)\, dt - \frac{n}{2}\int_{-1/n}^{1/n} g(0)\, dt\right| \\
&= \left|\frac{n}{2}\int_{-1/n}^{1/n} (g(t) - g(0))\, dt\right| \\
&\leq \frac{n}{2}\int_{-1/n}^{1/n} |g(t) - g(0)|\, dt \\
&< \frac{n}{2}\int_{-1/n}^{1/n} \epsilon = \epsilon.
\end{aligned}$$

The second to last inequality is due to the integral triangle inequality, and the final inequality is due to the fact that $t \in [-\frac{1}{n}, \frac{1}{n}]$. Thus $|t - 0| \leq \frac{1}{n} < \gamma$. This completes the proof. $\square$

Definition 1.6 (Dirac Delta Functional) *Let g be a continuous function on $\mathbb{R}$. Then $\delta(t)$ is the generalized function such that*

$$\int_{\mathbb{R}} g(t)\delta(t)\, dt = g(0).$$

The name Dirac delta functional comes from physicist Paul Dirac who applied the functional to quantum mechanics. Now, note that

$$(g * \delta)(t) = \int_{\mathbb{R}} g(t-\tau)\delta(\tau)\, d\tau = g(t-0) = g(t).$$

So δ is the identity under the convolution operation. To verify the other conditions, note that if $g(t) = 1$ then

$$\int_{\mathbb{R}} \delta(t)\, dt = \int_{\mathbb{R}} g(t)\delta(t) = g(0) = 1.$$

This verifies (i). Now, note that for any $t \neq 0$, there exists a $n \in \mathbb{N}$ such that $\frac{1}{n} < |t|$. Thus, $f_n(t) = 0$, verifying (ii). Also, each $f_n(t) \geq 0$ for all $n \in \mathbb{N},\ t \in \mathbb{R}$, verifying (iii).

In general we cannot interchange the limit and the integrand; this is again a heuristic argument. We are also interested in the δ functional with respect to the Fourier Transform. Let $\delta_{t_0} = \delta(t - t_0)$. Then

$$\widehat{\delta}(\omega) = \int_{\mathbb{R}} \delta(t) \exp^{-2\pi i t\omega}\, dt = \exp^{-2\pi i(0)\omega} = 1$$

$$\widehat{\delta_{t_0}}(\omega) = \int_{\mathbb{R}} \delta(t - t_0) \exp^{-2\pi i t\omega}\, dt = \exp^{-2\pi i t_0\omega}$$

and with respect to the inverse transform

$$\check{\delta}(t) = \int_{\mathbb{R}} \delta(\omega) \exp^{2\pi i t\omega}\, dt = \exp^{2\pi i t(0)} = 1$$

$$\check{\delta_{\omega_0}}(t) = \int_{\mathbb{R}} \delta(\omega - \omega_0) \exp^{-2\pi i t\omega}\, dt = \exp^{-2\pi i t\omega_0}.$$

To introduce the idea of band-limit, we shall use the *cosine* function, which has a well defined frequency. Using the definition of *cosine* on $\mathbb{C}$, note that

$$\cos(2\pi\alpha t) = \frac{1}{2}\left[\exp^{2\pi i t\alpha} - \exp^{-2\pi i t\alpha}\right] = \frac{1}{2}\left[\check{\delta}(\omega - \alpha) - \check{\delta}(\omega + \alpha)\right]$$

and taking the Fourier Transform, we get

$$\widehat{\cos}(2\pi\alpha t) = \widehat{\left[\frac{1}{2}\left[\check{\delta}(\omega - \alpha) - \check{\delta}(\omega + \alpha)\right]\right]} = \frac{1}{2}\left[\delta(\omega - \alpha) - \delta(\omega + \alpha)\right].$$

The last equality is 'justified' by the linearity of the transform and the fact that transform and the inverse transform cancel out, i.e. $\check{\widehat{f}} = \widehat{\check{f}} = f$ for all $f \in L^1(\mathbb{R})$. Again, this is a heuristic argument since *cosine* is not a $L^1(\mathbb{R})$ function. However, this allows us to observe that $\cos(2\pi\alpha t)$ is α band-limited, as it should be since α is the singular frequency of $\cos(2\pi\alpha t)$.

Definition 1.7 (Band-limited) *The function $f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ is called Ω-band-limited if the support of its Fourier transform $\widehat{f}$ is contained within the bounded interval $[-\Omega, \Omega]$ in $\widehat{\mathbb{R}}$,* *i.e.* $supp(\widehat{f}) \subset [-\Omega, \Omega]$.

Letting

$$f(t) = \cos(2\pi\alpha t)$$

and taking Fourier Transform using the earlier analysis, we arrive at

$$\widehat{f}(\omega) = \delta(\omega - \alpha) - \delta(\omega + \alpha).$$

Thus $\widehat{f}(\omega) = 0$ for all $\omega \notin [-\alpha, \alpha]$, as desired.

More rigorous development of generalized functions, and specifically the Dirac delta functional, can be found in [Erd62] and [Lig62].

1.4 Paley-Wiener Theorem

A compactly supported function $f(t)$ has a real analytic Fourier transform $\widehat{f}(\omega)$ that can be analytically continued to the entire complex plane $\mathbb{C}$. This continuation is the Fourier-Laplace transform of f defined by

$$\widehat{f}(\zeta) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \cdot \zeta}\, dt\ ,\ \zeta \in \mathbb{C}.$$

The Fourier transform can be thought of as the restriction of the Fourier-Laplace transform to the real axis. Further analysis of this transform gives way to the following theorem.

Theorem 1.1 (Paley-Wiener) *The Fourier-Laplace transform of an infinitely differentiable function f with compact support contained in $\{|t| \leq A\}$ is an entire function $\widehat{f}(\zeta)$ in $\mathbb{C}$ which satisfies the following property:*

For every integer $n \geq 0$ there exists a positive constant C_n such that

$$|\widehat{f}(\zeta)| \leq C_n (1 + |\zeta|)^{-n} \exp^{2\pi A |\Im(\zeta)|} \text{ for all } \zeta \in \mathbb{C}.$$

Conversely, every entire function in $\mathbb{C}$ satisfying this property is the Fourier-Laplace transform of a C^∞ function with compact support contained in $\{|t| \leq A\}$.

The classical Paley-Wiener theorem says that a square-integrable complex-valued function, defined over the real line, can be extended off the real line as an entire function of exponential type A if and only if its Fourier transform $F(\omega)$ is identically zero for $|\omega| > A$, i.e., if and only if F is band-limited to $[-A, A]$. (For a derivation, see [DM72].)

2 Classical Sampling Theorem

Sampling theory is the basis for the digital representation of analog signals. The original goal in this area of research was to convert a continuous analog signal into a discrete amount of data that completely represents the original signal, demonstrated in the following theorem.

2.1 Statement of the Theorem

Theorem 2.1 (Classical Sampling Theorem) *Let f be a continuous, Ω-band-limited function of finite energy on $\mathbb{R}$, i.e. $f \in L^2(\mathbb{R})$.*

i. If $T \leq 1/2\Omega$, then for all $t \in \mathbb{R}$,

$$f(t) = T \sum_{n \in \mathbb{Z}} f(nT) \frac{\sin(\frac{\pi}{T}(t - nT))}{\pi(t - nT)}.$$

ii. If $T \leq 1/2\Omega$ and $f(nT) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.

This theorem is at the heart of digital representation of an analog signals. It states that certain functions can be perfectly represented by sample values of the function taken at regular intervals. The requirement of a limit on the bandwidth of the signal is fundamental in determining the upper bound on the sample rate. The discovery of this rate is credited to Nyquist, and thus the rate is commonly referred to as the Nyquist rate [Nyq28]. Sampling at or above[5] this rate allows for complete reconstruction; sampling any faster does not add any information.

The history of the development of this theorem is fairly complex and includes a dispute over who initially discovered the result. The theorem has several names associated with it: the cardinal series, the Whittaker Sampling Theorem, the Kotelnikov Theorem, and

[5]'Above' refers to a faster rate.

the Shannon Sampling Theorem; here we shall simply refer to it as the classical sampling theorem as a distinction from the multi-rate sampling theorem discussed later in the paper.

2.2 Proof of the Theorem

The proof of this theorem can be found, in varying forms, in texts such as [Ben74] or [Hig96]. This proof is recreated from [Cas10].

Proof of Theorem 2.1: Let f be Ω-band-limited, i.e. $\text{supp}(\widehat{f}) \subset [-\Omega, \Omega]$. Then for all $n \in \mathbb{Z}$, let

$$g(u) = \begin{cases} \widehat{f}(u) & \text{if } u \in [-\Omega, \Omega] \\ \widehat{f}(u - 2\Omega n) & \text{if } u \in [2\Omega n - \Omega, 2\Omega n + \Omega]. \end{cases}$$

We call $g(u)$ the 2Ω-periodic extension of $\widehat{f}$ outside the support of $\widehat{f}$. Since g is 2Ω-periodic by construction, we can represent it by its Fourier Series (Def 1.2):

$$g(u) = \sum_{n \in \mathbb{Z}} c_n \exp^{-i\pi n u/\Omega}, \tag{4}$$

with coefficients c_n given by

$$\begin{aligned} c_n &= \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} g(u) \exp^{i\pi n u/\Omega} \, du \\ &= \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \widehat{f}(u) \exp^{i\pi n u/\Omega} \, du \\ &= \frac{1}{2\Omega} \int_{\mathbb{R}} \widehat{f}(u) \exp^{2i\pi (n/2\Omega) u} \, du \\ &= \frac{1}{2\Omega} f\left(\frac{n}{2\Omega}\right). \end{aligned}$$

The second equality is due to the fact that $\widehat{f} = g$ on $[-\Omega, \Omega]$. The third is due to the fact that $\widehat{f} = 0$ on $(-\infty, -\Omega) \cup (\Omega, \infty)$. The final equality is simply the recognition of the inverse

Fourier transform in the above line. Clearly, $\widehat{f} = g \cdot \chi_{[-\Omega,\Omega]}$. Substituting this into (4)

$$\begin{aligned}\widehat{f}(u) &= \sum_{n\in\mathbb{Z}} c_n \exp^{-i\pi nu/\Omega} \cdot \chi_{[-\Omega,\Omega]}(u) \\ &= \frac{1}{2\Omega}\sum_{n\in\mathbb{Z}} f\left(\frac{n}{2\Omega}\right) \exp^{-i\pi nu/\Omega} \cdot \chi_{[-\Omega,\Omega]}(u).\end{aligned}$$

The inverse Fourier transform and the uniform convergence of the Fourier series give way to the representation

$$\begin{aligned}f(x) &= \int_{\mathbb{R}} \widehat{f}(u) \exp^{2\pi iux}\, du \\ &= \int_{\mathbb{R}} \frac{1}{2\Omega}\sum_{n\in\mathbb{Z}} f(\frac{n}{2\Omega}) \exp^{-i\pi nu/\Omega} \cdot \chi_{[-\Omega,\Omega]} \exp^{2\pi iux}\, du \\ &= \frac{1}{2\Omega}\sum_{n\in\mathbb{Z}} f(\frac{n}{2\Omega}) \int_{\mathbb{R}} \exp^{-i\pi nu/\Omega} \cdot \chi_{[-\Omega,\Omega]} \exp^{2\pi iux}\, du \\ &= \frac{1}{2\Omega}\sum_{n\in\mathbb{Z}} f(\frac{n}{2\Omega}) \int_{-\Omega}^{\Omega} \exp^{(2\pi ix - \pi in/\Omega)u}\, du \\ &= \frac{1}{2\Omega}\sum_{n\in\mathbb{Z}} f(\frac{n}{2\Omega}) 2\Omega \frac{\sin\pi(2\Omega x - n)}{\pi(2\Omega x - n)} \\ &= \sum_{n\in\mathbb{Z}} f(\frac{n}{2\Omega}) \frac{\sin\pi(2\Omega x - n)}{\pi(2\Omega x - n)}.\end{aligned}$$

Due to uniform convergence, we can exchange the integral and summation in the third equality. The fourth equality follows from the nature of $\chi_{[-\Omega,\Omega]}$ Allowing $T = 1/2\Omega$, we have our desired result.

$$\begin{aligned}f(x) &= \sum_{n\in\mathbb{Z}} f(nT) \frac{\sin\pi(x/T - n)}{\pi(x/T - n)} \\ &= \sum_{n\in\mathbb{Z}} f(nT) \frac{\sin(\frac{\pi}{T}(x - nT))}{\frac{\pi}{T}(x - Tn)} \\ &= T\sum_{n\in\mathbb{Z}} f(nT) \frac{\sin(\frac{\pi}{T}(x - nT))}{\pi(x - nT)}.\end{aligned}$$

$\square$

2.3 Application to Audio Recording

Fundamentally, sound is an analog signal; waves travel through the molecules of air and are sensed by our ears. The first audio recording devices, such as Edison's cylinder phonograph, recorded the continuous amplitude of sound into a cylinder of wax. The phonograph played back the sound by measuring the amplitude recorded in the wax with a needle. Vinyl records were a direct extension of this technology, as were cassette tapes, which left a magnetic imprint, rather than a physical one.

Since human hearing is band limited, the classical sampling theorem can be applied to any audible sound. Human hearing cannot detect frequencies above 20 kHz. This does not guarantee that the actual sound is band limited, however. This disconnect between the physical phenomenon of sound and the mathematical construct of the classical sampling theorem can be resolved. Suppose we have a function which describes a sound, f, but is not band limited. We can construct a new function which only contains content below some frequency Ω. Let

$$\widehat{g} = \widehat{f} \cdot \chi_{[-\Omega,\Omega]}.$$

Taking the inverse Fourier transform $(\widehat{g})\check{} = g$, we arrive at a function which is band limited. If we do this for $\Omega = 20\text{kHz}$, then the human ear will be unable to distinguish between f and g. This is not a mathematical statement, but instead an assumption that audio engineers make when designing, building, and using microphones.

Any realizable microphone can detect frequencies in a limited range. This is equivalent to the above mathematical process and allows us to apply the classical sampling theorem. Sampling at 44.1 kHz is above the Nyquist rate for human hearing; this is the sampling rate of sound recorded on a CD. Digital audio recording measures the amplitude of a signal at sufficiently small, constant intervals. Typically the right and left channels are recorded separately; surround sound recording systems use up to 7.1 channels (the .1 referring to a separate bass channel for a subwoofer). We are not concerned with channels, as each one can be treated as an individual audio signal.

High definition audio is usually recorded at 96 kHz with each sample stored as 24 bits.

This rate is well above the Nyquist rate[6]. From this master recording, CDs are resampled to 44.1 kHz and each sample is reduced to 16 bits. Other formats, such as MP3s and the FLAC lossless codec, also exist. The process of resampling will be discussed later in Section 4.

[6]In fact it is roughly the typical engineering solution to working with a bound: double it and add 10.

3 Multi-Rate Sampling Theory

The classical sampling theorem shows that a bandwidth-limited function can be completely reconstructed by sampling at or above the Nyquist rate. If the sampling rate is slower, the function cannot be reconstructed using the classical theorem. Research in sub-Nyquist sampling has proven effective using other methods. The theorem presented here was discovered and proved by Casey and Walnut [CW94].

The purpose of the theorem is to develop a process of sampling at multiple, sub-Nyquist rates which together can reconstruct the signal completely, i.e. as if it were sampled above the Nyquist rate. The rates are required to be incommensurate for the theorem to hold. This condition guarantees that there is no overlapping data between the samples taken at different rates.

3.1 Multichannel Deconvolution

Deconvolution has proven itself to be a useful mathematical tool in the field of signal and image processing. For images, it acts as an enhancing filter to correct out-of-focus blurs in a picture. For signals, it can be used to correct distorted line shapes without a loss of signal-noise ratio (SNR). In essence, deconvolution is useful in areas of signal and image processing where the incoming data contains a high degree of information. The reason this works so naturally is because the convolution equation models a number of linear systems.

For our purposes, the convolution equation $s = f * \mu$ models linear, translation invariant systems (e.g. microphones). In this model, f is the input signal function, μ is the system impulse response distribution, and s is the output (received) signal. However, in many physical applications, s is often a poor approximation of the signal f. This motivates us to deconvolve f from μ to attain the original signal. Results have shown that if the convolver μ is time-limited (i.e. compactly supported) and non-singular (i.e. not a delta generalized function), then this problem is 'ill-posed' in the sense of Hadamard [Par89]. It has been shown to be ill-posed for all realizable convolvers, i.e. all convolvers that can be built. To circumvent this scenario, a theory of multichannel deconvolution has been developed to solve these equations. A multichannel system preserves information about the signal that would otherwise be lost. Thus, data lost by one convolver can still be retained by another convolver. Together, the signals s_i overdetermine f since

$$s_i = f * \mu_i, \ i = 1, \ldots, n.$$

If the convolvers $\{\mu_i\}$ satisfy the condition of being strongly coprime, deconvolving f is now well-posed.

Definition 3.1 (Strongly Coprime) *A set of convolvers $\{\mu_i\}$ that satisfy the inequality*

$$\left(\sum_{i=1}^{n} |\widehat{\mu}_i(\zeta)|^2\right)^{1/2} \geq A \exp^{-B|\Im z|} \cdot (1 + |\zeta|)^{-N}$$

for every $\zeta \in \mathbb{C}$, where A and B are positive constants, N is a positive integer and $\Im z$ denotes the imaginary part of z, is said to be strongly coprime.

Note that this tells us that $\frac{1}{\sum_{i=1}^{n} |\widehat{\mu}_i(\zeta)|^2}$ satisfies the Paley-Weiner growth bound. If this is so, then there exists a set of time-limited deconvolvers $\{\nu_i\}$ such that

$$\mu_1 * \nu_1 + \cdots + \mu_n * \nu_n = \delta$$

and, consequently

$$\widehat{\mu}_1 \cdot \widehat{\nu}_1 + \cdots + \widehat{\mu}_n \cdot \widehat{\nu}_n = 1, \tag{5}$$

where δ is the Dirac delta functional. Equation (5) above is the analytic Bezout equation. The existence of such deconvolvers is guaranteed by the following theorem:

Theorem 3.1 (Hörmander) *For compactly supported and strongly coprime distributions*

$\{\mu_i\}_{i=1}^n$ on $\mathbb{R}$, there exists compactly supported distributions $\{\nu_i\}_{i=1}^n$ such that

$$\delta = \mu_1 * \nu_1 + \cdots + \mu_n * \nu_n.$$

Moreover, the set of convolvers $\{u_i\}$ and deconvolovers $\{v_i\}$ satisfy the analytic Bezout equation (5) if and only if the set of distributions $\{\mu_i\}_{i=1}^n$ are strongly coprime.

By Hörmander's theorem, a strongly coprime set has a solution in the analytic Bezout equation. Thus, given the deconvolvers and output signals, f is naturally produced.

$$\sum_i s_i * \nu_i = \sum_i (f * \mu_i) * \nu_i = \sum_i f * (\mu_i * \nu_i) = f * \sum_i (\mu_i * \nu_i) = f * \delta = f.$$

The system is such that no information is lost in this process. This occurs because the condition of being strongly coprime guarantees that the zeros in the analytic Bezout equation do not cluster quickly as $|\zeta| \to \infty$. If the $\{\widehat{\mu}_i\}$ did have a common zero, then $\widehat{s}_i(\zeta) = 0$ at that zero and information about f would be lost. Because the system is engineered toward eliminating any common zero, no information about f is lost and the problem is well-posed. Thus, the signal f is gathered by this strongly coprime system, and these received signals are then filtered by the deconvolvers to reconstruct f.

These methods are linear and 'realizable,' thus deconvolution at some time sample only depends on the information near that time sample. Unfortunately, Hörmander's Theorem is an existence theorem; it does not reveal what the deconvolvers might be. One obvious solution is

$$\nu_i = \frac{\overline{\mu_i}}{\sum_i |\mu_i|^2}.$$

The deconvolvers have been shown not to be unique, and so, for certain scenarios one set of deconvolvers may be better than another. More in this direction is given in [CW94].

3.2 Non-Commensurate Sampling Lattices

The theory developed here merges the two ideas of multichannel deconvolution and classical sampling theory together. It has been developed by Casey [Cas07], Casey and

Sadler [CS00], and Walnut [Wal96].

By the results in multichannel deconvolution, we need to create a set of strongly coprime convolvers $\{\mu_i\}$ in order for the problem to be well-posed. First, we present three definitions. Then, the following theorem will be useful in creating such a set.

Definition 3.2 *A number $\alpha \in \mathbb{R}$ is said to be poorly approximated by rationals if there exists an integer $n \in \mathbb{N} : n \geq 2$ and a constant $C_\alpha > 0$ such that for all integers $p, q \in \mathbb{Z} : q \geq 2$,*

$$\left|\alpha - \left(\frac{p}{q}\right)\right| \geq \frac{C_\alpha}{q^n}.$$

This class of numbers will be denoted by $\mathbb{P}$.

Definition 3.3 *A number $\alpha \in (\mathbb{R}\backslash\mathbb{Q})$ is said to be well approximated by rationals if for all $n \in \mathbb{N} : n > 2$ and for all constants $C > 0$, there exists integers $p, q \in \mathbb{Z} : q \geq 2$,*

$$\left|\alpha - \left(\frac{p}{q}\right)\right| < \frac{C}{q^n}.$$

This class of numbers will be denoted by $\mathbb{W}$.

Definition 3.4 (Liouville numbers) *A number $\alpha \in (\mathbb{R}\backslash\mathbb{Q})$ is said to be a Liouville number if for all $n \in \mathbb{N} : n > 2$ there exists integers $p, q \in \mathbb{Z} : q \geq 2$,*

$$\left|\alpha - \left(\frac{p}{q}\right)\right| < \frac{1}{q^n}.$$

This class of numbers will be denoted by $\mathbb{L}$.

From Definition 3.2 and 3.3, it is clear that $\mathbb{P}$ is the complement of $\mathbb{W}$ with respect to $(\mathbb{R}\backslash\mathbb{Q})$. It is also clear from Definition 3.3 and 3.4 that $\mathbb{W} \subseteq \mathbb{L}$. It is established, as a exercise in the final chapter of [TBB01], that $\lambda(\mathbb{L}) = 0$, and thus $\lambda(\mathbb{W}) = 0$. Therefore $\mathbb{P}$ has full measure: it is the complement of a zero measure set with respect to a set of full measure. The proof that $\mathbb{L}$ is of measure zero uses the fact that

$$\mathbb{L} = (\mathbb{R}\backslash\mathbb{Q}) \cap \bigcap_{n=2}^{\infty} G_n$$

where G_n are open sets defined as

$$G_n := \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n} \right).$$

A little work yields the fact that this intersection can be covered by arbitrarily small intervals; thus, $\mathbb{L}$ has measure zero. The Golden Mean is a classic example of an irrational which is poorly approximated by rationals, however there are uncountably many. The (irrational) algebraic numbers are a subset of $\mathbb{P}$, and $\mathbb{P}$ is dense in $\mathbb{R}$. We shall use this class of numbers to construct strongly coprime sets.

Theorem 3.2 (Petersen-Middleton) *Let $0 < r_1 < \cdots < r_m$, $m \geq d = 1$ satisfy the condition that for all $i \neq j$, r_i/r_j are in $\mathbb{P}$, then $\{\chi_{[-r_i,r_i]^d}\}$ is a strongly coprime set.*

Then to create a set of appropriate convolvers, we need only find such a set whose ratios are poorly approximated by rationals.

Suppose $f \in L^2(\mathbb{R})$ and let $\alpha \in \mathbb{P}$. Now the pair $\mu_1 = \chi_{[-1,1]}$, $\mu_2 = \chi_{[-\alpha,\alpha]}$ are strongly coprime. Thus, the problem of solving the analytic Bezout equation for ν_1 and ν_2

$$\widehat{\mu}_1 \cdot \widehat{\nu}_1 + \widehat{\mu}_2 \cdot \widehat{\nu}_2 = 1$$

is well-posed. Now, $\widehat{\mu}_1(\omega) = \frac{\sin(2\pi\omega)}{\pi\omega}$ and $\widehat{\mu}_2(\omega) = \frac{\sin(2\pi\alpha\omega)}{\pi\omega}$ have zeros[7]

$$Z_{\widehat{\mu}_1} = \left\{ \frac{\pm k}{2} : k \in \mathbb{N} \right\},\ Z_{\widehat{\mu}_2} = \left\{ \frac{\pm k}{2\alpha} : k \in \mathbb{N} \right\}.$$

Thus, one solution to the analytic Bezout equation is $\widehat{\nu}_1(\omega) = \frac{1}{2\widehat{\mu}_1(\omega)}$ for $\omega \in Z_{\widehat{\mu}_1}$ and, likewise, $\widehat{\nu}_2(\omega) = \frac{1}{2\widehat{\mu}_2(\omega)}$ for $\omega \in Z_{\widehat{\mu}_2}$. Then, the problem simply becomes an interpolation problem on the set of zeros $\Gamma = Z_{\widehat{\mu}_1} \bigcup Z_{\widehat{\mu}_2}$. We then have a new reconstruction formula for

[7]Here and throughout, $0 \notin \mathbb{N}$.

f from its values on Γ. and the following theorem:

Theorem 3.3 (Casey-Walnut) *Let*

$$\Gamma = \left\{ \frac{\pm k}{2} : k \in \mathbb{N} \right\} \bigcup \left\{ \frac{\pm k}{2\alpha} : k \in \mathbb{N} \right\}$$

and let $\lambda_i \in \Gamma$, $\alpha \in \mathbb{P}$. *If* f *is a* $(1+\alpha)$*-band-limited function, then* f *can be conditionally reconstructed by the formula*

$$f(t) = \sum_{\lambda \in \Gamma} f(\lambda) \frac{G(t)}{G\prime(\lambda)(t-\lambda)} + [f(0)K_1(t) + f'(0)K_2(t)]$$

where

$$\begin{aligned} G(t) &= \sin(2\pi t) \cdot \sin(2\pi \alpha t) \\ K_1(t) &= \frac{G(t)}{\frac{G''(0)}{2!}t^2} \\ K_2(t) &= \frac{G(t)}{\frac{G''(0)}{2!}t}. \end{aligned}$$

The interpolators at the origin appropriately model the function at the origin, i.e. $K_1(0) = 1$ while $K_1'(0) = 0$ and $K_2(0) = 0$ while $K_2'(0) = 1$. Also, note that the information contained in the signal can be reconstructed uniquely by sampling at each $\lambda \in \Gamma \cup \{0\}$ [Wal96]. The signal is conditionally reconstructed with respect to conditions on specialized projections into Hilbert subspaces. More on this can be found in [Cas07] and [CW94]. Here, it is also important to note that the sampling rates correspond to 1-band-limited functions and α-band-limited functions. 1 and α add up to the band-limit of the function to be sampled. Thus, we have here a reconstruction of a $(1+\alpha)$-band-limited function via significantly lower sampling rates.

This result generalizes for an arbitrary number of sampling rates. Let $\{r_i\}_{i=1}^{n}$ be a strongly coprime set. Then, our convolvers have the form $\mu_i(t) = \chi_{[-r_i, r_i]}(t)$. These $\{\mu_i\}$

model the impulse response of a multichannel system and have Fourier transforms

$$\widehat{\mu}_i(\zeta) = \frac{\sin(2\pi r_i \zeta)}{\pi\zeta}$$

with zeros sets $Z_i = \{\frac{\pm k}{2r_i} : k \in \mathbb{N}\}$. Note that exclusive of the origin, the zeros sets are nonrepetitive. Let $\Gamma_i = Z_i$ and $\Gamma = \bigcup_{i=1}^{n} \Gamma_i$. Thus, we can reconstruct f on its values at Γ. This can also be done via techniques in complex interpolation theory.

Thus, we have the following theorem:

Theorem 3.4 (Casey-Walnut) *Let $\{r_i\}_{i=1}^n$ be a strongly coprime set, and let f be a $(\sum r_i)$-band-limited function. Let*

$$\Gamma_i = \left\{ \frac{\pm k}{2r_i} : k \in \mathbb{N} \right\}$$

for $k \in \mathbb{N}$ and $i = 1, \dots, n$. Also, let

$$\Gamma = \bigcup_{i=1}^{n} \Gamma_i.$$

Then f is uniquely determined by

$$\{f(\lambda) : \lambda \in \Gamma\} \bigcup \{f(0), \dots, f^{n-1}(0)\}.$$

Furthermore, f can be conditionally reconstructed from its values on $\Gamma \bigcup \{0\}$ by the following formula

$$f(t) = \sum_{\lambda_i \in \Gamma} f(\lambda_i) \frac{G(t)}{G'(\lambda_i)(t - \lambda_i)} + \sum_{i=1}^{n} f^{(i-1)}(0) \cdot K_i(0)$$

where

$$G(t) = \prod_{i=1}^{n} \sin(2\pi r_i t)$$

and the interpolating functions $K_i(t)$ at the origin are a linear combination of $\frac{G(t)}{t^j}$, $j = 1, \dots, n$, chosen so that $K_i^{(l-1)}(0) = \delta_{i,l}$, $i, l = 1, \dots, n$.

3.3 Applications of the Theorem

As mentioned in the introduction, the multi-rate theorem improved techniques measuring the flatness of a silicon chip. Casey and Walnut developed this theorem to simulate a sensor that was more precise than any physically realizable sensor. There are other useful applications to ultra-wideband signal detection, specifically with respect to communication and radio telescope technology.

This paper develops an application of the theorem to audio signal representation. The aim is to sample audio at multiple, incommensurate, sub-Nyquist rates. In theory, the audio signal can be perfectly reconstructed from union of the samples taken at each rate. It can also be estimated from a subset of the samples. Currently, when audio is streamed over a network, it is done in a single file at constant quality. If the bandwidth of the network is too low for the quality, a new lower-quality file must be loaded. This is not a seamless process and causes a break in playback. Using the multi-rate sampling theorem, we develop strategy for a streaming music system which can change quality on the fly.

4 Digital to Analog Conversion

The previous sections have dealt with analog to digital conversion, i.e., transforming a continuous, band-limited, time signal into a discrete set of data. This is useful for digital storage of these types of signals, specifically audio files. However, when a sound is played from through a speaker, it receives an analog signal. Thus, the computer, specifically the sound card, takes the discrete audio samples, along with the sampling rate, and reconstructs an analog signal which is electrically feed to the speaker. While many high end audio cards can work on a range of sampling rates, audio is typically sampled at 44.1kHz.

4.1 Constructing the Analog Signal

At first it seems that the analog signal can be accessed directly using the classical sampling theorem. Suppose we have an audio signal which is sampled at a rate $T \leq 1/2\Omega$, i.e. $f(nT)$ is known for all $n \in \mathbb{Z}$. Then the audio signal is given by

$$f(t) = T \sum_{n \in \mathbb{Z}} f(nT) \frac{\sin(\frac{\pi}{T}(t - nT))}{\pi(t - nT)}.$$

A few things are immediately clear. Our audio file, for obvious reasons, is not of infinite length, and if it were, it would be computationally impossible to compute this infinite sum. For a sound signal of finite length, we can attempt to compute this sum over each samples, however in practice this is impractical. The sum generates round off errors over time and, since an appropriate sampling rate yields thousands of samples a second, the reconstruction ultimately fails. Thus, we turn to other forms of interpolation to construct the analog signal.

4.2 Polynomial Interpolation

Given a set of $k+1$ data points $(t_1, f(t_1)), ..., (t_{k+1}, f(t_{k+1}))$ we can construct a k^{th} degree polynomial whose graph intersects each point in the set [BF05].

Definition 4.1 (Lagrange Interpolating Polynomial) *Let* $(t_1, f(t_1)), ..., (t_{k+1}, f(t_{k+1}))$ *be* $k+1$ *distinct known values of a function. Then*

$$p(t) := \sum_{j=1}^{k+1} f(t_j) l_j(t)$$

where

$$l_j(t) := \prod_{i=1, i\neq j}^{k+1} \frac{t - t_i}{t_j - t_i} = \frac{t - t_1}{t_j - t_1} \cdots \frac{t - t_{j-1}}{t_j - t_{j-1}} \frac{t - t_{j+1}}{t_j - t_{j+1}} \cdots \frac{t - t_{k+1}}{t_j - t_{k+1}}.$$

Note that for $t = t_j$ and $m \neq j$

$$l_j(t_j) = \prod_{i=1, i\neq j}^{k+1} \frac{t_j - t_i}{t_j - t_i} = \prod_{i=1, i\neq j}^{k+1} 1 = 1,$$

$$l_m(t_j) = \prod_{i=1, i\neq m}^{k+1} \frac{t_j - t_i}{t_m - t_i} = \left(\frac{t_j - t_j}{t_m - t_j}\right) \prod_{i=1, i\neq m, i\neq j}^{k+1} \frac{t_j - t_i}{t_m - t_i} = (0) \prod_{i=1, i\neq m, i\neq j}^{k+1} \frac{t_j - t_i}{t_m - t_i} = 0.$$

Therefore

$$p(t_j) = \sum_{i=1}^{k+1} f(t_i) l_i(t) = f(t_j) l_j(t_j) + \sum_{i=1, i\neq j}^{k+1} f(t_i) l_i(t) = f(t_j)(1) + \sum_{i=1}^{k+1} f(t_i)(0) = f(t_j)$$

as desired. That is, the interpolating polynomial $p(t)$ is equivalent to the function $f(t)$ at each of the $k+1$ points.

The difference between the function and the interpolating polynomial is bounded if you have knowledge of the $k+1$ derivative and it is bounded. Suppose each $t_i \in [a, b]$. Then for any $t \in [a, b]$ there exists a $\xi_t \in (a, b)$ such that

$$f(t) = p(t) + \frac{f^{(k+1)}(\xi_t)}{(k+1)!} \prod_{i=1}^{k+1} (t - t_i).$$

This agrees with the above analysis showing that $f(t_i) = p(t_i)$ for each t_i, since

$$\prod_{j=1}^{k+1}(t_i - t_j) = (t_i - t_i)\prod_{j=1;j\neq i}^{k+1}(t_i - t_j) = 0 \cdot \prod_{j=1;j\neq i}^{k+1}(t_i - t_j) = 0.$$

Just as in the case of the classical sampling formula, this polynomial is excessively computationally demanding to interpolated over every sample point in the audio file. Moreover, interpolating with high degree polynomials suffer from Runge's phenomenon: extreme oscillation resulting in a large error term. Thus, we construct a distinct polynomial $p_{t_0}(t)$ for each t_0 we wish to resample at using only the close points in our data set $t_i \in (t_0 - \epsilon, t_0 + \epsilon)$. We then evaluate $p_{t_0}(t_0)$ to approximate the value of $f(t_0)$. Each of these polynomials is called a 'spline'; the process of using cubic splines is the above process using four interpolation points (and thus a third degree, or cubic, polynomial). Splines are typically cut and pasted together to construct a new continuous interpolating function. However, we are not concerned with the construction of such a function, but only the function value at an unknown time t_0.

4.3 Sample Rate Conversion

The process of sample rate conversion has been heavily studied. The audio tracks on digital video are typically recorded at 48kHz, while digital audio is typically recorded at 44.1kHz or 96kHz. Assuming the audio recorded at 96kHz has no content with frequencies above 24kHz, we can simply remove every other point. The classical sampling theorem provides the motivation for this and shows that if we sample down past the Nyquist rate, we will not be able to reconstruct the signal perfectly.

Suppose we wanted to convert to a new rate, r_1, which is not simply half the initial rate, r_0. If the two rates have a rational ratio, then we sample up to their least common multiple, $\text{lcm}(r_0, r_1)$. This could be done by interpolation, however in practice it is not. Regardless of how it is done, this signal will contain frequencies above $\frac{r_1}{2}$. A digital low pass filter is then used to remove all frequencies above $\frac{r_1}{2}$. We can then take every k^{th} sample, where $k = \frac{\text{lcm}(r_0,r_1)}{r_1}$, and we will have a file which is sampled at our new rate r_1.

I did not extensively research this method since, by construction, we choose rates which do not have rational ratios. However, the method above does not resample to move up to the $\mathrm{lcm}(r_0, r_1)$ rate, but instead just adds the value of 0 for all the unknown samples. The application of the digital low pass filter acts similarly to an interpolator, adjusting the values of each sample. Also, each of these sample values does not need to be computed, only the k^{th} value that is needed for the r_1 rate. Since these processes are not directly applicable, further discussion in this area is not presented here. More information in these processes can be found in [Lyo04] and [CR83]. Further research in these types of filters may help improve and refine the process described in the following section.

4.4 Audio Card Hardware

In a computer, sounds are stored as digital data. An audio card receives this digital data, the sample points and the corresponding rates, and outputs a continuous electric signal which connects to a speaker. Before the development of this technology, computers were restricted to MIDI sounds. Rather than taking in sample values, and the rate which they were sampled at, the computer told the audio card which note to play. Middle A on a piano is described by $\sin((440)2\pi t)$, for example. This however, limits the signals that can be played to combinations of basic musical notes. Clearly this technology is not sufficient for playing recorded sound.

The ideal audio card would take these samples recorded above the Nyquist rate, interpolate the original audio signal, and output an electric signal to a speaker. Figure 1 illustrates a signal with samples taken at regular intervals. The function is the ideal signal we aim to reconstruct. However, in practice, the signal is not interpolated by the audio card. Most sound cards use a process called a 'zero-order hold', illustrated in Figure 2. The audio card simply outputs the sample value over the sample interval, resulting in a piecewise constant (discontinuous) function. Slightly more advanced systems use a 'first-order hold', which constructs a piecewise linear (continuous) function which connects each sample value. The illustration in Figure 3 is an example of this process, however it is not typically used in hardware. It is also important to note that when we resample at incommensurate rates, we require a higher degree of accuracy then either of these methods.

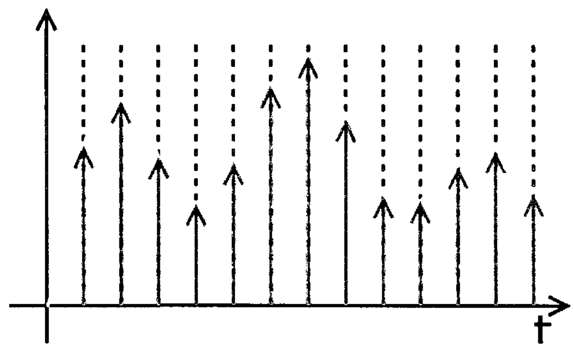

Figure 1: Ideal Digital to Analog Conversion [Pet06b]

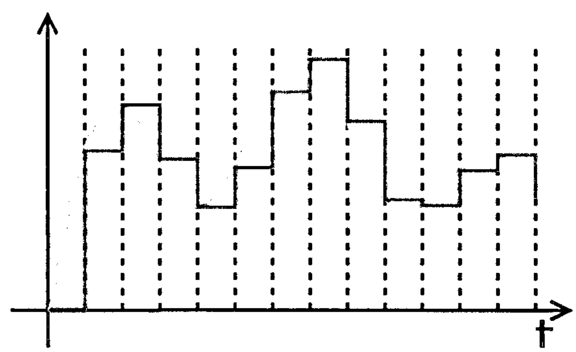

Figure 2: Zero-Order Hold [Pet06c]

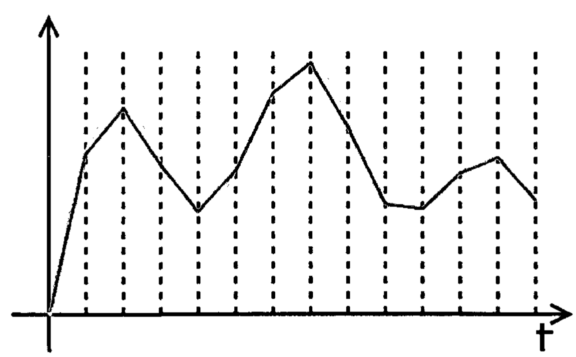

Figure 3: First-Order Hold [Pet06a]

5 Program Structure

5.1 Architecture

These flow charts outline the architecture of the program on the server and client side. The strategy is to use resampling techniques to generate three new files, each at new incommensurate rates, from the original file. These files are then streamed individually, with different priority, over a broadband channel or cell phone network. The client receiving the stream will then re-interpolate back at the Nyquist rate from all the files, or a subset, depending on which have loaded sufficiently. This re-interpolated file is then sent to the computer's audio card. Figure 4 outlines the server side function, and Figure 5 outlines the client side function.

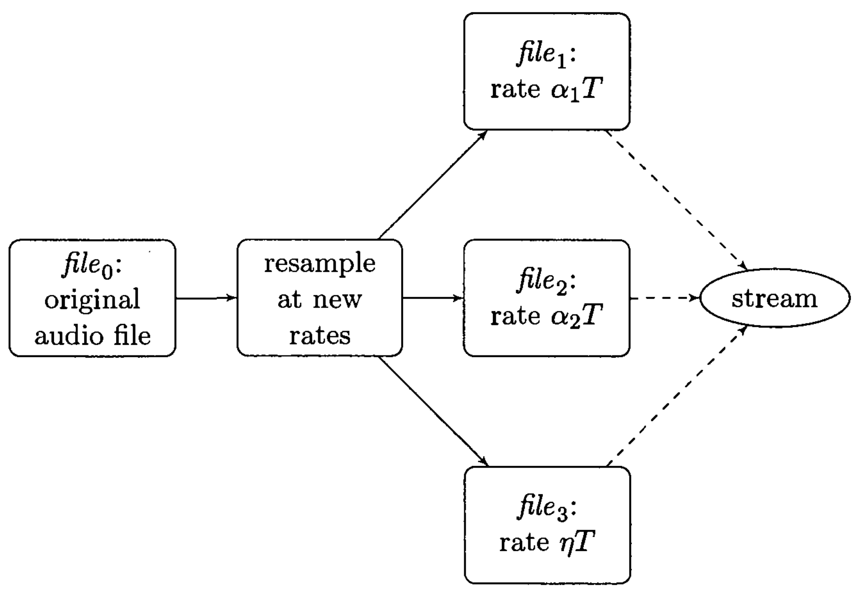

Figure 4: Server Side Programing Outline

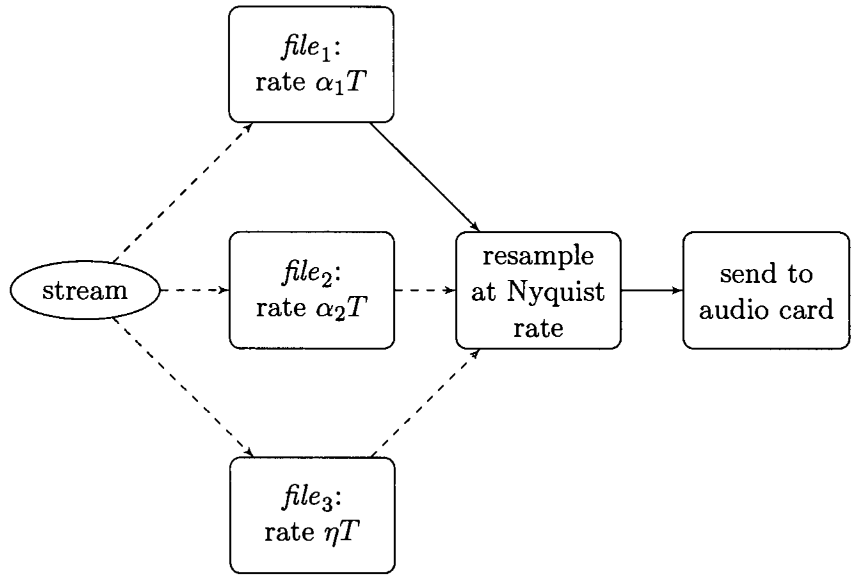

Figure 5: Client Side Programing Outline

5.2 Resampling at Incommensurate Rates

The process of resampling is not trivial; due to the complexity of the classical sampling formula, other methods of approximation are used. As described in Section 4.2, constructing low degree polynomials from near by known values may be an appropriate method. Recall the definition (4.1) of the Lagrange interpolating polynomial.

Suppose we want to resample a file of length $\bar{t}$ at the new rate αT. Then we wish to resample at the time values $n\alpha T$ for each $n \in \mathbb{N}$ such that $n\alpha T \leq \bar{t}$, where T is the original above Nyquist sampling rate. So let $t_0 = n_0 \alpha T$ for the n_0 we are currently attempting to resample at. The value $f(mT)$ is known for all $mT \leq \bar{t}$, $m \in \mathbb{N}$. We want to collect τ points near t_0, so we will create an auxiliary look up function which will return a vector, $\mathbf{m}= (m_1, m_2, ..., m_\tau)$. These vectors will be initially computed so that $m_1 T$ is the closest point to t_0, $m_2 T$ is the second closest point, and so on. Figure 6 outlines the process of estimating $f(t)$ at each t_0.

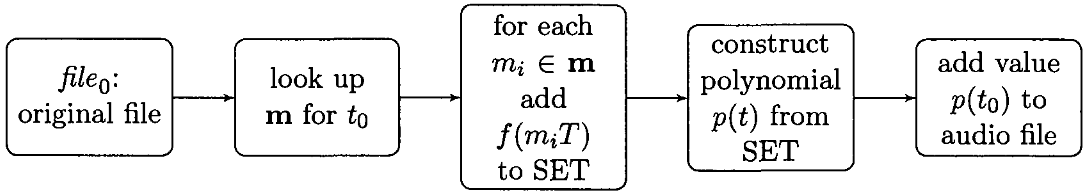

Figure 6: Resampling Algorithm

5.3 Choosing Incommensurate Rates

There is a fundamental issue with using an incommensurate sampling rate numerically. Each rate α from Theorems 3.3 and 3.4 must be an irrational which is poorly approximated by rationals. Since every number in a computer is rational by the physical reality of computation, we cannot satisfy the conditions of the theorems. However, in practice representing these numbers with 64-bits seems to have sufficient results. Recall again Theorem 3.4, which will motivate the following theorem. A $(1+\alpha_1+\alpha_2)$ bandwidth limited function with $\alpha_1, \alpha_2, \frac{\alpha_1}{\alpha_2} \in \mathbb{P}$ is completely determined by the function values on the set

$$\Gamma = \left\{\frac{\pm k}{2} : k \in \mathbb{N}\right\} \bigcup \left\{\frac{\pm k}{2\alpha_1} : k \in \mathbb{N}\right\} \bigcup \left\{\frac{\pm k}{2\alpha_2} : k \in \mathbb{N}\right\}.$$

Theorem 5.1 *Suppose $f(t)$ is an Ω-band limited function and let $T = \frac{1}{2\Omega}$. If $\eta \in \mathbb{N}$, α_1, α_2, and their ratios are poorly approximated by rationals, and*

$$\Omega \leq \frac{\Omega}{\eta} + \frac{\Omega}{\alpha_1} + \frac{\Omega}{\alpha_2}. \tag{6}$$

Then $f(t)$ is completely determined on the set

$$\Gamma \quad = \quad \{\pm k\eta T : k \in \mathbb{N}\} \bigcup \{\pm k\alpha_1 T : k \in \mathbb{N}\} \bigcup \{\pm k\alpha_2 T : k \in \mathbb{N}\}.$$

Proof of Theorem 5.1: Applying Theorem 3.4, $f(t)$ is completely determined on the set

$$\begin{aligned}\Gamma &= \left\{\frac{\pm k}{2(\Omega/\eta)} : k \in \mathbb{N}\right\} \bigcup \left\{\frac{\pm k}{2(\Omega/\alpha_1)} : k \in \mathbb{N}\right\} \bigcup \left\{\frac{\pm k}{2(\Omega/\alpha_2)} : k \in \mathbb{N}\right\} \\ &= \left\{\frac{\pm k\eta}{2\Omega} : k \in \mathbb{N}\right\} \bigcup \left\{\frac{\pm k\alpha_1}{2\Omega} : k \in \mathbb{N}\right\} \bigcup \left\{\frac{\pm k\alpha_2}{2\Omega} : k \in \mathbb{N}\right\}.\end{aligned}$$

Since we have chosen $T = \frac{1}{2\Omega}$, substituting completes the proof.

$$\Gamma = \{\pm k\eta T : k \in \mathbb{N}\} \bigcup \{\pm k\alpha_1 T : k \in \mathbb{N}\} \bigcup \{\pm k\alpha_2 T : k \in \mathbb{N}\}$$

□

Now, dividing Equation (6) by Ω we arrive at

$$1 \leq \frac{1}{\eta} + \frac{1}{\alpha_1} + \frac{1}{\alpha_2} \tag{7}$$

which is useful in choosing η, α_1, α_2. We are not limited to using three rates, but we choose three now for simplicity. The preceding theorem extends to n incommensurate rates in the obvious way. To obtain the files sampled at rates $\alpha_1 T, \alpha_2 T$, we will have to resample using the procedure in the previous section. However, the file which is sampled at ηT is simply a subset of the original audio file. The process of resampling has an error term which we attempt to minimize, but selecting a subset of audio sample is exact. Also, when the file is reconstructed, each of the points in this file correspond exact with some of the points we reconstruct. Thus, the lower η, the more sample points that are kept without error and the faster the client side reconstruction.

We shall use the notation used in Figure 4 and Figure 5.

$$\begin{aligned} file_0 &= (\{kT \in \mathbb{N}\}, f(\{kT\})) \\ file_1 &= (\{k\alpha_1 T\}, f(\{k\alpha_1 T\})) \\ file_2 &= (\{k\alpha_2 T\}, f(\{k\alpha_2 T\})) \\ file_3 &= (\{k\eta T\}, f(\{k\eta T\})) \end{aligned}$$

Here we omit $k \in \mathbb{N}$ from the set notation since in each file, k will reach a distinct maximum determined by the length of the original file. If $\eta = 1$, then we simply have the original file, $file_0$. Thus, we shall choose $\eta = 2$. Then $file_3$ will contain half the data in $file_0$. The choice of α_1 and α_2 rely mainly on Equation (7), however we also wish include more data in $file_1$, so that if $file_2$ lags behind during the streaming, most of the can be reconstructed from $file_3$ and $file_1$. Let $\alpha_1 = \frac{9}{5}\phi$ and $\alpha_2 = 3\sqrt{3}$, where ϕ is the Golden mean. Then a quick numerical calculation shows Equation (7) is satisfied.

$$\begin{aligned} \frac{1}{\alpha_1} &\approx 0.34335 \\ \frac{1}{\alpha_2} &\approx 0.19245 \\ \frac{1}{\eta} + \frac{1}{\alpha_1} + \frac{1}{\alpha_2} &\approx 1.03580 \\ \frac{1}{\eta} + \frac{1}{\alpha_1} &\approx 0.84335 \end{aligned}$$

Also note that about 84.3% of the data is contained in just $file_1$ and $file_3$.

5.4 Resampling for Playback

Once we have created $file_1$, $file_2$, and $file_3$, we must develop a process for reconstructing and playing the audio. We aim to allow the quality to change as the bandwidth fluctuates. This paper does not address the technicalities of steaming data over a network and simply assumes that available bandwidth can be allocated differently to each stream. We shall give highest priority to $file_3$, since it both contains most of the data and requires far less processing for playback. The next highest priority is given to $file_1$ and the lowest to $file_2$, agreeing with the relative data stored in each.

The software will resample the audio signal at the original rate T, finding the values of $f(t_0)$ at values $t_0 = kT$ for appropriate $k \in \mathbb{N}$ with respect to the length of the audio file. These values will be put into a new file, $file_r$. We will again be using a Lagrange interpolating polynomial discussed in Section 4.2 to estimate the signal function $f(t)$, unless the points are already contained in $file_3$. Every other point we wish to find will be in $file_3$ and directly added to *file*$_r$. Figure 7 outlines the process for estimating $f(t_0)$ for $t_0 \notin$ *file*$_3$.

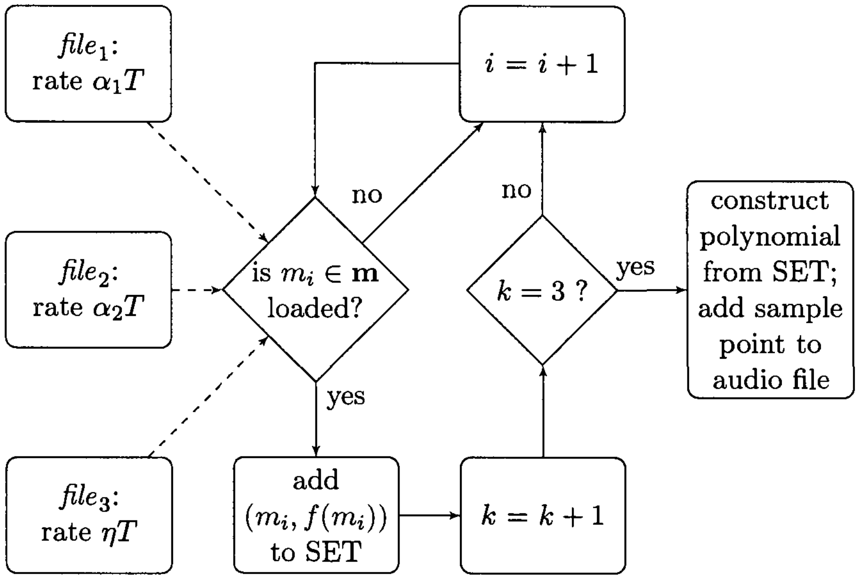

Figure 7: Playback Algorithm

Playback Algorithm. Playback requires resampling the signal at the Nyquist Rate. This flowchart describes determining the value of the signal at an unknown value t_0 while the signal is being played at $\vec{t}$.

6 Programing in Mathematica$_{®}$6

Our overall programming strategy is to take a file which is sampled at or above Nyquist, reconstruct the analog signal, and resample this function at our incommensurate rates. These multiple data sets, each with a corresponding rate, will comprise the new audio file. Since the audio hardware takes a single rate, the software which will play the audio file will use the data from each of these new file and re-interpolate at the Nyquist rate of 44.1kHz.

6.1 'Interpolation' Function

The 'Interpolation' in Mathematica$_{®}$6 fits polynomial curves between data points. By default, these polynomials are of third degree, however a higher (or lower) order can be specified. This is exactly the process described in Section 4.2. Deciding which order to use depends on desired accuracy and available computational time. If we assume that the initial construction of $file_1$ and $file_2$ will be done once and stored, accuracy is more important than computational time. However, on the client side, processing power may be limited, and this interpolation must be computed faster than the audio is played. Thus computation is the more important factor in the process described above in Figure 7.

6.2 Generating New Data Points

The following code was used in Mathematica$_{®}$6 to generate the new files, $file_1$, $file_2$, and $file_3$, resampled at incommensurate rates. The rates used were

$$2T;\ \frac{9\phi}{5}T;\ 3\sqrt{3}T$$

for $\phi =$ Golden Mean, $T = 44.1$kHz as discussed in Section 5.3.

```
SetDirectory["PATH_TO_FILE"]

Import["original.aif", "Elements"]

MasterList = Import["original.aif", "Data"]

LeftMasterList = MasterList[[1]]

RightMasterList = MasterList[[2]]

Rate = Import["original.aif", "SampleRate"]

Size = Length[LeftMasterList]

SampleTimes = Range[0, Size/Rate - 1/Rate, 1/Rate]

LeftSamplePoints = Transpose[{SampleTimes, LeftMasterList}]

RightSamplePoints = Transpose[{SampleTimes, RightMasterList}]

LeftTestFunction =

 Interpolation[LeftSamplePoints, InterpolationOrder -> 7]

RightTestFunction =

 Interpolation[RightSamplePoints, InterpolationOrder -> 7]

a = N[(9/5) GoldenRatio]

a2 = N[3*Sqrt[3]]

n = 2

aSampleTimes = N[Range[0, Size/Rate, a/Rate]]

aLeftSampleList = LeftTestFunction[aSampleTimes]

aLeftSamplePoints = Transpose[{aSampleTimes, aLeftSampleList}]

aLeftSamplePoints2 = Delete[aLeftSamplePoints, 1]

aRightSampleList = RightTestFunction[aSampleTimes]

aRightSamplePoints = Transpose[{aSampleTimes, aRightSampleList}]

aRightSamplePoints2 = Delete[aRightSamplePoints, 1]

a2SampleTimes = N[Range[0, Size/Rate, a2/Rate]]

a2LeftSampleList = LeftTestFunction[a2SampleTimes]

a2LeftSamplePoints =  Transpose[{a2SampleTimes, a2LeftSampleList}]

a2LeftSamplePoints2 = Delete[a2LeftSamplePoints, 1]
```

```
a2RightSampleList = RightTestFunction[a2SampleTimes]
a2RightSamplePoints =  Transpose[{a2SampleTimes, a2RightSampleList}]
a2RightSamplePoints2 = Delete[a2RightSamplePoints, 1]
nSampleTimes = Range[0, Size/Rate - 1/Rate, n/Rate]
nLeftSampleList = LeftMasterList[[1 ;; Size ;; n]]
nLeftSamplePoints = Transpose[{nSampleTimes, nLeftSampleList}]
nRightSampleList = RightMasterList[[1 ;; Size ;; n]]
nRightSamplePoints = Transpose[{nSampleTimes, nRightSampleList}]
aSampleList = {aLeftSampleList, aRightSampleList}
a2SampleList = {a2LeftSampleList, a2RightSampleList}
nSampleList = {nLeftSampleList, nRightSampleList}
ReconstructedList = {LeftReconstructedList, RightReconstructedList}
Export["testalpha1.aif",  ListPlay[aSampleList,
 SampleRate -> Round[N[Rate/a], 1]]]
Export["testalpha2.aif", ListPlay[a2SampleList,
 SampleRate -> Round[N[Rate/a2], 1]]]
Export["testn.aif", ListPlay[nSampleList,
 SampleRate -> Round[N[Rate/n], 1]]]
```

6.3 Combining Layered Data

The following code was used to reconstruct two new files. The first is reconstructed from $file_1$, $file_2$ and $file_3$, and the second was reconstructed only from $file_1$ and $file_3$. The purpose of creating this file from only $file_1$ and $file_3$ was to examine the quality of the sound in the context of streaming. This file is an example of the quality when the bandwidth drops and $file_2$ drops out of the stream.

```
LeftJoinedList =
 Join[nLeftSamplePoints, aLeftSamplePoints2, a2LeftSamplePoints2]
LeftJoinFunction = Interpolation[LeftJoinedList]
```

```
LeftReconstructedList = LeftJoinFunction[SampleTimes]
RightJoinedList = Join[nRightSamplePoints, aRightSamplePoints2,
 a2RightSamplePoints2]
RightJoinFunction = Interpolation[RightJoinedList]
RightReconstructedList = RightJoinFunction[SampleTimes]
ReconstructedList = {LeftReconstructedList, RightReconstructedList}
Export["reconstruct.aif", ListPlay[ReconstructedList, SampleRate -> Rate]]
LeftJoinedList1 = Join[nLeftSamplePoints, aLeftSamplePoints2]
LeftJoinFunction1 = Interpolation[LeftJoinedList1]
LeftReconstructedList1 = LeftJoinFunction1[SampleTimes]
RightJoinedList1 = Join[nRightSamplePoints, aRightSamplePoints2]
RightJoinFunction1 = Interpolation[RightJoinedList1]
RightReconstructedList1 = RightJoinFunction1[SampleTimes]
ReconstructedList1 = {LeftReconstructedList1, RightReconstructedList1}
Export["part_reconstruct.aif", ListPlay[ReconstructedList1,
 SampleRate -> Rate]]
```

7 Conclusions

7.1 Programing in Python

The next step in this research is to develop it in a more robust programing environment. The Python programming language offers a suitable environment to do much of this computation. Python has an extended mathematics library, as well as an import support for audio files. There are a number of open source projects for audio compression.

Ideally, this algorithm would be incorporated into a compression algorithm, optimizing the playback process. This is critical to the utility of the algorithm: a compressed audio file is far smaller than even half the original size. Broadcasting uncompressed audio files would require much more bandwidth than the entire compressed file.

7.2 Other Similar Processes

While completing this research, I listened to a streaming audio service on my cell phone. The audio at the very beginning, maybe half a second of audio, of the first song played usually starts at a low quality and quickly improves. This is done without stopping and loading a new stream, but simply changing quality on the fly. I do not know what algorithm was used using, however I doubt they are using the process I have invented. This motivated me to think of other possible processes which would allow for quality to change during playback.

The simplest process I could construct breaks an audio file into 2 audio files, by removing every other sample value from the original file and placing it into new file. These files can be streamed separately, with different bandwidth priority. If both files load, audio plays back as normal; if only one loads, the file plays back at the same quality as $file_3$ introduced in Section 5. This process could be extended to by breaking the files up into any of the following parts without resampling: $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$, $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\}$, and so on.

Such a system has the advantage of simplicity. There is also no interpolation and thus no loss in data when all streams load. This saves processing power on both the server and client side. However using incommensurate sampling rates allows for much more control over the percentage of information stored in each stream. Studies on peoples ability to distinguish the difference between these two processes, and their preferences, would be useful in comparing the given utility between the two. One process may also be more useful with respect to the compression algorithms, which usually rely on eliminating redundancies in the frequency domains.

7.3 Error Analysis

The following tables, Figure 8 and Figure 9, give some of the statistics on the reconstructed files. The statistics are run on the difference between the original data points and the reconstructed data points. The Mathematica code below gives the exact calculations. The most illuminating statistic is the ratio between the mean of the original points, and the mean of the difference. This is can be thought of as the proportion of error, however this is not an exact process.

```
PreDifference = MasterList[[1]] - ReconstructedList[[1]]
Difference = PreDifference^2
Max = Max[Difference]
Mean = Mean[Difference]
Standard Deviation = StandardDeviation[Difference]
MasterMean = Mean[MasterList[[1]]^2]
Ratio = Mean/MasterMean
```

7.4 Application to Video

The advancement of higher speed cellular internet connections may soon make this entire process unnecessary due to the relatively small size of audio files. However, there may be an application to video. A video file is represented by a three-dimensional matrix

Reconstructed File	Max	$\bar{x}$	σ_x	Ratio
Example 1	1.115×10^{-1}	1.054×10^{-5}	2.565×10^{-4}	4.486×10^{-4}
Example 2	1.491×10^{-3}	1.057×10^{-7}	3.552×10^{-6}	1.515×10^{-5}
Example 3	9.520×10^{-4}	7.350×10^{-8}	1.840×10^{-6}	1.134×10^{-5}
Example 4	1.011×10^{0}	2.825×10^{-5}	1.702×10^{-3}	6.015×10^{-4}

Figure 8: Statistics on Reconstructed Files

Partially Reconstructed	Max	$\bar{x}$	σ_x	Ratio
Example 1	3.948×10^{-2}	1.622×10^{-5}	2.065×10^{-4}	6.902×10^{-4}
Example 2	1.876×10^{-4}	1.557×10^{-6}	1.396×10^{-6}	2.231×10^{-5}
Example 3	6.704×10^{-6}	1.231×10^{-7}	7.260×10^{-7}	1.900×10^{-5}
Example 4	2.596×10^{-2}	4.099×10^{-5}	8.728×10^{-4}	2.119×10^{-4}

Figure 9: Statistics on Partially Reconstructed Files

composed of color information. Each entry in the matrix is a pixel; the first two dimensions spans the height and width of video frames, and the third dimension spans the frames over time. Current compression algorithms exploit redundancies over both the spacial and time information.

Since the light our eyes can see is bandwidth limited, between 400-790 THz[8], the classical sampling theorem should apply, as well as the multi-rate sampling theorem. However, currently the fastest frame rate commercially used in high definition video is around 60 frames per second, clearly far less than $2\Omega = 2 \cdot (790\text{THz}) = 1580$ THz, i.e. 158×10^{13} frames per second. The relationship between sampling theory and video may not be quite as obvious as with audio, at least with respect to time. It could, however, be quite useful over the spatial dimensions of the video. The multi-rate sampling theorem could also be a useful tool to increase the maximum frame rate currently possible by using multiple cameras recording at incommensurate frame rates. Further research in this area and collaboration with video engineers could reach interesting results.

[8]THz is short for terahertz, 10^{12} Hz.

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