SUPERCUSPIDAL CHARACTERS OF REDUCTIVE p-ADIC GROUPS

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ABSTRACT. We compute the characters of many supercuspidal representations of reductive p-adic groups. Specifically, we deal with representations that arise via Yu's construction from data satisfying a certain compactness condition. Each character is expressed in terms of a depth-zero character of a smaller group, the (linear) characters appearing in Yu's construction, Fourier transforms of orbital integrals, and certain signs and cardinalities that are described explicitly in terms of the datum associated to the representation and of the element at which the character is evaluated.

CONTENTS

0. INTRODUCTION

0.1. **History.** Suppose F is a non-Archimedean local field, $\mathbf{G}$ is a connected reductive F-group, and $G = \mathbf{G}(F)$. For simplicity of the present discussion, assume that F has characteristic zero. For π an irreducible, admissible representation of G, let Θ_π denote the distribution character of π, a linear functional on the space $C_c^\infty(G)$ of locally constant, compactly supported functions on G. Howe [35] and Harish-Chandra [29] showed that Θ_π can be represented by a locally constant function on the set of regular, semisimple elements of G. We will also denote the representing function by Θ_π.

A great deal is known about the asymptotic behavior of characters (as functions) near singular points. For example, the blow-up of Θ_π is controlled by the fact, due to Harish-Chandra [30], that $|D_G|^{1/2} \Theta_\pi$ is locally integrable on G, where D_G is a certain polynomial function on G, the *discriminant* of G. From Howe [33] and Harish-Chandra [30], we know that, near a singular point, Θ_π (composed with a suitable logarithmic map) has an expansion in terms of Fourier transforms of nilpotent orbital integrals. More recent work has made precise where these expansions hold (see [27, 45] for a conjecture, [21] for the main result, and [5] for a generalization); presented other expansions, where the collection of orbital integrals involved is smaller and depends on π (see [47–50]); or done both (see [4, 18, 23, 37, 38]).

Despite the work mentioned above, in most cases, we do not have explicit character formulas, even in a limited domain, because neither the orbital integrals nor their coefficients are understood explicitly (though see [7, 20, 55] for exceptions). In practice, such formulas usually arise from explicit information about the construction of representations. However, the construction methods can be quite complicated.

Let us restrict our attention to supercuspidal representations. Suppose that the residual characteristic p of F is odd. Then earlier work has yielded character formulas for all of the supercuspidal representations of SL_2 [53] (using the construction in [52], which is known to be exhaustive by [54]); PGL_2 [60]; GL_2 [59]; GL_ℓ [17, 19]; SL_ℓ [61]; and division algebras of degree ℓ [16, 17]. In the latter cases, ℓ is a prime that is sufficiently small with respect to p. In addition, one knows the characters of many depth-zero representations of unramified groups [22] (namely, those induced from inflations of Deligne–Lusztig representations of associated finite groups of Lie type), certain depth-zero character values for Sp_4 [8], and inductive formulas for characters of division algebras [15].

An earlier announcement [3] contains formulas for the characters of (necessarily supercuspidal) representations of the multiplicative group of a central division algebra over a p-adic field. The present paper generalizes these

results to the setting of general tame reductive groups over a p-adic field of odd residual characteristic, where the construction of J.-K. Yu (see [65] and our §2) can be used to replace that of Corwin, Howe, and Moy (see, for example, [14, 34, 44]). If p is large enough, then all supercuspidals of G arise via this construction (see [39]). If $\mathbf{G}$ is GL_n or the multiplicative group of a central division algebra over F of index n, then the Corwin–Howe–Moy construction is known to be exhaustive even if we assume only that p does not divide n (see [44]). In [26, §3.5], Hakim and Murnaghan discuss the relationship between this construction and Yu's.

There are other constructions of supercuspidal representations that make no tameness assumptions. These start with [40, 41] and presently culminate in [9–11, 57, 63]. However, an attempt to use these to compute explicit character formulas would require a different approach in order to overcome many serious technical difficulties. For example, among many other things, we make use of Bruhat–Tits theory and Moy–Prasad filtrations, both of which behave poorly under wild Galois descent.

0.2. **Outline of this paper.** In order to evaluate the character Θ_π of a representation π at a regular, semisimple element γ in G, we require first of all that γ lie near a tame F-torus. If p is larger than a constant determined by the root system of $\mathbf{G}$, then all semisimple elements of G have this property. Second, we require that γ be well approximated by a product of good elements. Such approximations, called "normal r-approximations", are analogous to truncations of expressions of elements of $F^\times$ in the form

$$\varepsilon^{m_0}\varpi^d \cdot \prod_{i=1}^{\infty}(1 + \varepsilon^{m_i}\varpi^i),$$

where ϖ is a uniformizer of F and ε is a root of unity in F of order coprime to p. From Lemma 8.1 of [6], we see that many tame elements of G have such an expansion. Under mild hypotheses, which are always satisfied when $\mathbf{G}$ is an inner form of GL_n, all tame elements of $\mathbf{G}$ have such an expansion. The expansions we require, together with their basic properties, are discussed in [6]. The reader may find it particularly convenient to have at hand the statements of Lemmata 5.29 and 5.32, Proposition 5.40, and Remarks 6.7 and 6.10 of *loc. cit.* (Analogous approximations, with analogous properties, exist for elements of the Lie algebra of G. The proofs are similar to, but easier than, those in *loc. cit.*)

After presenting our basic notation in §1, we outline (in §2) Yu's construction of supercuspidal representations (see [65]). Briefly, Yu starts with a sequence $(\mathbf{G}^0 \subsetneq \cdots \subsetneq \mathbf{G}^d = \mathbf{G})$ of reductive groups, together with (an inducing datum for) a depth-zero, supercuspidal representation π_0' of G^0, a character ϕ_i of each G^i, and a point $\overline{x}$ in the reduced building of $\mathbf{G}^0$ over

F, all satisfying certain properties. He then constructs inductively, for each $i = 0, \ldots, d$, a smooth representation ρ_i' of an open compact modulo center subgroup of G^i such that the representation π_i of G^i induced from $\rho_i' \otimes \phi_i$ is irreducible and supercuspidal.

We will assume for the remainder of this subsection that $d > 0$ and $\mathbf{G}^{d-1}/Z(\mathbf{G})$ is F-anisotropic. (Notice that the latter hypothesis follows from the former if $\mathbf{G}$ is F-anisotropic, or if $\mathbf{G}$ is GL_ℓ or SL_ℓ with ℓ a prime.) The reason for this assumption is that we require very precise control over the behavior, with respect to a given Moy–Prasad filtration, of certain commutators (see, for example, Propositions 4.3 and 5.3.2). Since the computation of the character of π requires the evaluation of an integral formula (see (6.5)) involving arbitrary conjugates of the element in which we are interested, we cannot guarantee this good behavior for arbitrary groups $\mathbf{G}^{d-1}$; but it *does* occur when our compactness condition is satisfied (see Corollary 4.5). Even without the compactness condition, we can still compute the character values at many points of a representation $\tau = \tau_d$ induced from $\rho_d' \otimes \phi_d$ to a large open compact modulo center subgroup of G (see §2). (In our situation, the representation τ_i of §2 is equal to π_i for $0 \leq i < d$.)

Since Weil representations over finite fields play an essential role in Yu's construction of supercuspidal representations, in §3 we compute some of their characters at certain elements, following Gérardin (see [25]).

After the Weil representation computations, our character computations broadly follow the strategies pursued in [17] and [19], both of which rely on vanishing results to cut down the support of the relevant characters. In [17], these vanishing results are approached by computing first not the full induced character, but rather the character of a representation induced to a smaller open and compact modulo center subgroup. For us, this is the representation σ_i defined in §2. The desired vanishing results are discussed in §4, where we use the fact that the character of σ_i transforms by a linear character near the identity (see Corollary 4.6) to prove Proposition 4.3

In §5, we compute the character of σ_i, using the results of §4 to cut down the class of elements we must consider. Although Proposition 5.3.3 is the result that is used most often in the sequel, the heart of this section is really Proposition 5.3.2. The proof of this result involves fairly intricate manipulations of the Frobenius formula (see [56]), based on our detailed understanding of the behavior of taking commutators with an element γ (see [6, §7]). Historically, supercuspidal character formulas (specifically, Theorem 4.2(c) of [17], Theorem 5.3.2 of [19], and Theorem 4(2) of [3]) have involved Gauss sums in some form. These sums also appear in the present setting, but in disguise. We devote §5.2 to recognising and computing them.

With vanishing results in place for this partially induced representation, it becomes easier to describe the character of the full induced representation.

After proving a few results (Lemmata 6.1–6.3) to show that certain naturally arising integrals converge, we compute the full character in Theorem 6.4 using Harish-Chandra's integral formula.

The construction of [65] is inductive, in the sense that an inducing datum for π_i is constructed from an inducing datum for π_{i-1} and some additional data. This means that, in order to compute the character of $\pi = \pi_d$, we focus on explicating the relationship between the characters of π_{d-1} and π_d (or, in the notation of §2, τ_{d-1} and τ_d). Theorem 6.4 is actually a statement about this relationship. Accordingly, the groups $\mathbf{G}^i$ and representations π_i for $0 \leq i < d - 1$ play no explicit role in our calculations until §7, where we unroll the inductive computations of §§3–6 to obtain an explicit formula for π_d in terms of the original datum.

The result of this unrolling is contained in Theorem 7.1. Applying the inductive formulas of the preceding sections, we obtain there a formula for $\Theta_\pi = \Theta_{\pi_d}$ in terms of the character of π'_0, the (linear) characters ϕ_i, and Fourier transforms of certain orbital integrals. (If $\mathbf{G}^{d-1}/Z(\mathbf{G})$ is not F-anisotropic, then we compute instead the character of $\tau = \tau_d$ in terms of essentially the same data, but with ρ'_0 in place of π'_0.) Also appearing in the character formulas are some explicitly defined positive constants (the numbers $c(\vec{\phi}, \gamma'_{<r})$ of Theorem 7.1) — essentially the square roots of cardinalities of certain quotients of filtration subgroups of G — and signs $\varepsilon(\phi, \gamma)$ and $\mathfrak{G}(\phi, \gamma)$ — computed in Propositions 3.8 and 5.2.13 in terms of the root system of $\mathbf{G}$ and various fields associated to the representation π and to the element γ.

Thus, we obtain formulas for evaluating, at many points, many supercuspidal characters of many groups.

0.3. **Future goals.** Our hypotheses are weak enough that, in the case of "tame" division algebras, i.e., those of index coprime to p, we can evaluate all characters at all points. In this case, the presence of considerable additional structural information (and fine control over conjugacy, thanks to the Skolem–Noether theorem) allows us to make the formula of Theorem 7.1 more explicit. In the process, we will correct an error in Theorem 4 of [3] (some of whose corollaries remain valid). This will be carried out in future work.

Work of Henniart (see [31]) has suggested that it is valuable to be able to recognise a representation given only the values of its character in a certain domain. Theorem 7.1 may be sufficiently explicit to allow us to describe a domain for which this can be done (at least, if we restrict ourselves to appropriate supercuspidal representations). This would nicely complement [26, Chapter 6], which describes another way of identifying supercuspidal representations.

The stability calculations of [22] proceed from Proposition 10.1.1 of *loc. cit.*, a "reduction formula". We have modelled our Theorem 6.4 after this reduction formula, and believe that the similarity of statements should provide a guide to stability calculations for positive-depth supercuspidal representations.

1. NOTATION AND PRELIMINARIES

1.1. **Generalities on fields and linear reductive groups.** Let $\widetilde{\mathbb{R}} = \mathbb{R} \sqcup \left\{ r+ \mid r \in \mathbb{R} \right\} \sqcup \{\infty\}$, and extend the order and additive structures on $\mathbb{R}$ to ones on $\widetilde{\mathbb{R}}$ in the usual way (see, for example, [6, §3.1]). For $\mathbb{F}$ a finite field, let $\mathrm{sgn}_{\mathbb{F}}$ denote the character of $\mathbb{F}^\times$ with kernel precisely $(\mathbb{F}^\times)^2$.

If F is a valued field with valuation ord, and $r \in \widetilde{\mathbb{R}}_{\geq 0}$, then let F_r denote $\left\{ t \in F \mid \mathrm{ord}(t) \geq r \right\}$. Then the residue field $\mathfrak{f} = \mathfrak{f}_F$ of F is the quotient F_0/F_{0+} of F_0 by F_{0+}. We will identify functions on F_0 that are trivial on F_{0+} with functions on $\mathfrak{f}$. In particular, if $\mathfrak{f}$ is finite, then we have the function $\mathrm{sgn}_{\mathfrak{f}}$ on F_0.

From now on, assume that F is locally compact, and that the characteristic p of its residue field $\mathfrak{f} = \mathfrak{f}_F$ is not 2. Fix an algebraic closure $\overline{F}$ of F, and let F^{un} and F^{sep} denote the maximal unramified and separable extensions of F in $\overline{F}$. Since F is Henselian, there is a unique extension of ord to each algebraic extension E/F (in particular, to $\overline{F}/F$), which we will denote again by ord.

Fix a square root $\sqrt{-1}$ of -1 in $\mathbb{C}$, and an additive character Λ of $\overline{F}$ that is trivial on $\overline{F}_{0+}$ and induces on $\mathfrak{f} = F_0/F_{0+}$ the character $t \mapsto \exp\left(2\pi\sqrt{-1}\,\mathrm{tr}_{\mathfrak{f}/\mathbb{F}_p}(t)/p\right)$, where $\mathbb{F}_p$ is the finite field with p elements. We may, and do, write again Λ for the resulting character of $\mathfrak{f}_E$, for any discretely valued algebraic extension E/F. Except in §5, we will be concerned only with the restriction to F of Λ. All Fourier transforms will be taken with respect to Λ. The particular

choice of square root will be of interest only in the statement of Proposition 5.2.13.

If we denote an algebraic F-group by a bold, capital, Latin letter, such as $\mathbf{H}$, then we will sometimes denote its Lie algebra by the corresponding bold, small Gothic letter, such as $\boldsymbol{\mathfrak{h}}$. We will denote sets of rational points by the corresponding non-bold letters, such as H and $\mathfrak{h}$.

For any set $S \subseteq X$, we denote by $[S]$ the characteristic function of S. (The "universe" X will be understood from the context.) If S is finite, then we denote by $|S|$ its cardinality. If $H' \subseteq H$ are groups and f is a function on $H'\backslash H$, then $\sum_{g \in H'\backslash H} f(g)$ will be shorthand for $\sum_{H'g \in H'\backslash H} f(H'g)$. Similar notation will be used for sums over double coset spaces.

Let $\mathbf{G}$ denote a reductive F-group. For the moment, we do not assume that $\mathbf{G}$ is connected. Let $\mathbf{G}^\circ$ denote the identity component of $\mathbf{G}$. Write $\mathfrak{g}^*$ for the dual Lie algebra of G, i.e., the vector-space dual of $\mathfrak{g}$.

Suppose that $X^* \in \mathfrak{g}^*$ is semisimple, in the sense that it is fixed by the coadjoint action of some maximal torus in $\mathbf{G}$. Any G-equivariant identification of $\mathfrak{g}^*$ with $\mathfrak{g}$ carries X^* to a semisimple element of $\mathfrak{g}$ (in the usual sense), so one knows that $G/C_G(X^*)$ carries an invariant measure, say $d\dot{g}$, and that the integral

$$\int_{G/C_G(X^*)} f(^gX^*)d\dot{g}$$

converges for $f \in C_c^\infty(\mathfrak{g}^*)$. Thus we may define a distribution $\hat{\mu}_{X^*}$ on $\mathfrak{g}$ by

$$\hat{\mu}_{X^*}(f) = \int_{G/C_G(X^*)} \hat{f}(^gX^*)d\dot{g} \quad \text{for } f \in C_c^\infty(\mathfrak{g}).$$

By Theorem A.1.2 of [4], $\hat{\mu}_{X^*}$ is representable by a locally constant function on the regular semisimple set in $\mathfrak{g}$. We may, and do, sometimes regard $\hat{\mu}_{X^*}$ as defined by an integral over G/Z, where Z is any closed cocompact unimodular subgroup of $C_G(X^*)$. By abuse of notation, we will denote again by $\hat{\mu}_{X^*}$ (or $\hat{\mu}_{X^*}^G$, if we wish to emphasize the ambient group G) the representing function. Notice that this function depends on the measure chosen.

If $\mathbf{M}$ is a Levi (not necessarily F-Levi) subgroup of $\mathbf{G}$, then, as in [65, §8], we identify the dual Lie algebras of $Z(\mathbf{M})$ and $\mathbf{M}$ with the fixed points in the dual Lie algebra of $\mathbf{G}$ for the coadjoint actions of $\mathbf{M}$ and $Z(\mathbf{M})$, respectively.

1.2. **Hypotheses.** Assume now, and for the remainder of the paper, that $\mathbf{G}$ is connected, splits over some tame extension of F, and satisfies Hypotheses (**B**) and (**C**) of [6]. By Remark 2.2 of *loc. cit.*, Hypotheses (**A**) and (**D**) follow from the tameness of $\mathbf{G}$. Thus, we may apply all the results of [6]. In some places, we also assume Hypothesis 2.3.

1.3. **Buildings and filtrations.** For any algebraic extension E/F of finite ramification degree, let $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, E)$ and $\mathcal{B}(\mathbf{G}, E)$ denote the reduced and enlarged Bruhat–Tits buildings of $\mathbf{G}(E)$, respectively. Then $\mathcal{B}(\mathbf{G}, E)$ is the product of $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, E)$ and an affine space. For a point $x \in \mathcal{B}(\mathbf{G}, E)$, let $\overline{x}$ denote the image of x under the natural projection to $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, E)$.

If H is a closed subgroup of G and $x \in \mathcal{B}(\mathbf{G}, F)$, then we will abbreviate $H \cap \mathrm{stab}_G(\overline{x})$ to $\mathrm{stab}_H(\overline{x})$. Note that, in this notation, $\overline{x}$ is an element of $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, F)$, not $\mathcal{B}^{\mathrm{red}}(\mathbf{H}, F)$, even if $H = \mathbf{H}(F)$ with $\mathbf{H}$ a compatibly filtered F-subgroup of $\mathbf{G}$ (as in Definition 4.3 of [6]) and $x \in \mathcal{B}(\mathbf{H}, F)$. Of course, if further $Z(\mathbf{H})/Z(\mathbf{G})$ is F-anisotropic, then actually there is no ambiguity, since we may regard $\mathcal{B}^{\mathrm{red}}(\mathbf{H}, F)$ as a subcomplex of $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, F)$.

Suppose $\mathbf{T}$ is a maximal F-torus in $\mathbf{G}$. Then we let $\Phi(\mathbf{G}, \mathbf{T})$ denote the set of roots of $\mathbf{T}$ in $\mathbf{G}$, and put $\widetilde{\Phi}(\mathbf{G}, \mathbf{T}) = \Phi(\mathbf{G}, \mathbf{T}) \cup \{0\}$. For each root $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, let $\mathrm{Lie}(\mathbf{G})_\alpha$ and $\mathbf{U}_\alpha$ denote the corresponding root space and root group, respectively. If $\alpha = 0$, then put $\mathrm{Lie}(\mathbf{G})_\alpha = \mathrm{Lie}(\mathbf{T})$ and $\mathbf{U}_\alpha = \mathbf{T}$. If $\mathbf{T}$ is F-split, then there are associated to $\mathbf{T}$ an affine space $\mathcal{A}(\mathbf{T})$ under $\mathbf{X}_*(\mathbf{T}) \otimes_\mathbb{Z} \mathbb{R}$, the lattice of cocharacters of $\mathbf{T}$ (tensored with $\mathbb{R}$), and an embedding of $\mathcal{A}(\mathbf{T})$ in $\mathcal{B}(\mathbf{T}, F)$. For us, an *affine root* will be either an affine function ψ on $\mathcal{A}(\mathbf{T})$ whose gradient $\dot{\psi}$ belongs to $\widetilde{\Phi}(\mathbf{G}, \mathbf{T})$, or a function of the form $\psi+\colon x \mapsto \psi(x)+$ with ψ as above. For each affine root ψ, we have a compact subgroup $_FU_\psi$ of $U_{\dot{\psi}}$ and a lattice $_F\mathfrak{u}_\psi$ in $\mathfrak{g}_{\dot{\psi}}$. Note that other authors reserve the term "affine root" for an affine function ψ such that $\dot{\psi} \in \Phi(\mathbf{G}, \mathbf{T})$ and $_FU_\psi \neq {}_FU_{\psi+}$.

In [45, §§2.6, 3.2] and [46, §§3.2–3], Moy and Prasad have defined, for each $x \in \mathcal{B}(\mathbf{G}, F)$, filtrations $(G_{x,r})_{r \in \mathbb{R}_{\geq 0}}$, $(\mathfrak{g}_{x,r})_{r \in \mathbb{R}}$, and $(\mathfrak{g}^*_{x,r})_{r \in \mathbb{R}}$ of G by compact open subgroups, $\mathfrak{g}$ by lattices, and $\mathfrak{g}^*$ by lattices, respectively. We extend these filtrations in the usual fashion to be defined for all $r \in \widetilde{\mathbb{R}}$ (or $r \in \widetilde{\mathbb{R}}_{\geq 0}$, in the case of the filtration on G). If $x \in \mathcal{B}(\mathbf{G}, F)$ and $g \in G_{x,0}$, then we let $\mathrm{d}_x(g)$ be the greatest index t such that $g \in G_{x,t}$. We define similar functions on $\mathfrak{g}$ and $\mathfrak{g}^*$ (not just $\mathfrak{g}_{x,0}$ and $\mathfrak{g}^*_{x,0}$), and denote them also by d_x.

If a group $\mathcal{G}$ has a filtration $(\mathcal{G}_i)_{i \in I}$, then we shall frequently write $\mathcal{G}_{i:j}$ in place of $\mathcal{G}_i/\mathcal{G}_j$ when $\mathcal{G}_j \subseteq \mathcal{G}_i$ (even if the quotient is not a group). For example, we put $F_{r:t} = F_r/F_t$, $U_{\psi_1:\psi_2} = U_{\psi_1}/U_{\psi_2}$, and $G_{x,r:t} = G_{x,r}/G_{x,t}$ for $r \leq t$ (and $r \geq 0$, in the last case) and for affine roots ψ_1 and ψ_2 such that $\dot{\psi}_1 = \dot{\psi}_2$ and $\psi_1 \leq \psi_2$.

By Proposition A.8, for each finite, tamely ramified extension E/F, tamely ramified maximal F-torus $\mathbf{T}$, and point $x \in \mathcal{B}(\mathbf{G}, E)$ (respectively, $x \in \mathcal{B}(\mathbf{G}, F)$), we have maps $\mathsf{e}^E_{x,t:u}$ and $\mathsf{e}_{\mathbf{T},x}$ satisfying Hypotheses A.1 and A.7. We write $\mathsf{e}_{x,t:u}$ for $\mathsf{e}^F_{x,t:u}$. If the choice of $\mathbf{T}$ is unimportant, then we will sometimes write e_x for $\mathsf{e}_{\mathbf{T},x}$.

In Definition 5.14 of [6], following [65, §§1–2], we defined, for $\mathbf{T}$ a tame maximal F-torus, filtration subgroups $_{\mathbf{T}}G_{x,f}$ of G associated to a pair (x, f) consisting of a point $x \in \mathcal{B}(\mathbf{T}, F)$ and a $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-invariant concave function f on the root system of $\mathbf{T}$ in $\mathbf{G}$ (see Definition 5.7 of [6]). It will be convenient here to define filtration lattices $_{\mathbf{T}}\mathrm{Lie}(G)_{x,f}$ in $\mathfrak{g}$ in the analogous fashion, by

$$_{\mathbf{T}}\mathrm{Lie}(G)_{x,f} := \sum {}_E\mathfrak{u}_\psi \cap \mathfrak{g},$$

where E/F is a tame splitting field for $\mathbf{T}$, the sum is taken over those affine roots ψ of $\mathbf{T}$ in $\mathbf{G}$ with $\psi(x) \geq f(\dot\psi)$. (In Definition 5.14 of [6], we had to take considerable care — for example, intersecting with some $G_{x',0}$ instead of just with G, since parahorics tend to behave badly under ramified descent; but, since Lie algebra filtrations are considerably better behaved (see, for example, Proposition 1.4.1 of [1]), such care is not necessary here.)

If $(\mathbf{T}, \vec{\mathbf{G}})$ is a tame reductive F-sequence in $\mathbf{G}$, in the sense of Definition 5.1 of [6], and $\vec{r}$ is an admissible sequence, in the sense of Definition 5.8 of *loc. cit.* (with associated concave function $f_{\vec{\mathbf{G}},\vec{r}}$), then, by analogy with the definition $\vec{G}_{x,\vec{r}} = {}_{\mathbf{T}}G_{x,f_{\vec{\mathbf{G}},\vec{r}}}$ of Definition 5.14 of *loc. cit.*, we put $\mathrm{Lie}(\vec{G})_{x,\vec{r}} = {}_{\mathbf{T}}\mathrm{Lie}(G)_{x,f_{\vec{\mathbf{G}},\vec{r}}}$. It is shown in Lemma 5.20 of *loc. cit.* that $\vec{G}_{x,\vec{r}}$ is independent of the choice of torus $\mathbf{T}$. The proof of the analogous result for $\mathrm{Lie}(\vec{G})_{x,\vec{r}}$ is, except for minor changes, the same. For convenience, by abuse of notation, we will often write $\mathrm{Lie}(_{\mathbf{T}}G_{x,f})$ in place of $_{\mathbf{T}}\mathrm{Lie}(G)_{x,f}$ and $\mathrm{Lie}(\vec{G}_{x,\vec{r}})$ in place of $\mathrm{Lie}(\vec{G})_{x,\vec{r}}$.

1.4. **Normal approximations.** We now define some basic concepts that will be needed in what follows. Since the definitions do not necessarily give the full flavor of what is going on, we give a "pictorial" example in Example 1.4.2 and describe a detailed computation in §1.5.

If $t \in \widetilde{\mathbb{R}}$ and $\underline{\gamma} = (\gamma_i)_{0 \leq i < t}$ is a good sequence in G (in the sense of Definition 6.4 of [6]), then put

$$C_{\mathbf{G}}^{(t)}(\underline{\gamma}) = \left(\bigcap_{0 \leq i < t} C_{\mathbf{G}}(\gamma_i) \right)^{\circ},$$

and

$$C_G^{(t)}(\underline{\gamma}) = C_{\mathbf{G}}^{(t)}(\underline{\gamma})(F).$$

In particular, $C_{\mathbf{G}}^{(t)}(\underline{\gamma}) = \mathbf{G}$ if $t \leq 0$. Note that the intersection defining $C_{\mathbf{G}}^{(t)}(\underline{\gamma})$ is really a *finite* intersection if $t < \infty$ (and, if $t = \infty$, then we have that $C_{\mathbf{G}}^{(\infty)}(\underline{\gamma}) = C_{\mathbf{G}}^{(t')}(\underline{\gamma})$ for $t' \in \mathbb{R}$ sufficiently large). We say (as in Definition 6.8 of *loc. cit.*) that $\underline{\gamma}$ is a *normal t-approximation* to an

element $\gamma \in G$ if there is an element $x \in \mathcal{B}(C_{\mathbf{G}}^{(t)}(\underline{\gamma}), F)$ such that $\gamma \in \left(\prod_{0 \le i < t} \gamma_i\right) C_G^{(t)}(\underline{\gamma})_{x,t}$. Sometimes, we will say for emphasis that $(\underline{\gamma}, x)$ is a normal t-approximation. In this case, we put

$$C_{\mathbf{G}}^{(t)}(\gamma) = C_{\mathbf{G}}^{(t)}(\underline{\gamma})$$
$$C_G^{(t)}(\gamma) = C_G^{(t)}(\underline{\gamma}),$$
$$Z_{\mathbf{G}}^{(t)}(\gamma) = Z(C_{\mathbf{G}}^{(t)}(\gamma)),$$

and

$$Z_G^{(t)}(\gamma) = Z_{\mathbf{G}}^{(t)}(\gamma)(F).$$

By Proposition 8.4 of *loc. cit.*, these groups are all independent of the choice of normal t-approximation to γ. Note that, if $t' \in \widetilde{\mathbb{R}}$ and $t' \le t$, then $\underline{\gamma}$ is also a normal t'-approximation to γ, so the notations $C_{\mathbf{G}}^{(t')}(\gamma)$ and $Z_{\mathbf{G}}^{(t')}(\gamma)$ are defined; and we have that $Z_{\mathbf{G}}^{(t')}(\gamma) \subseteq Z_{\mathbf{G}}^{(t)}(\gamma) \subseteq C_{\mathbf{G}}^{(t)}(\gamma) \subseteq C_{\mathbf{G}}^{(t')}(\gamma)$.
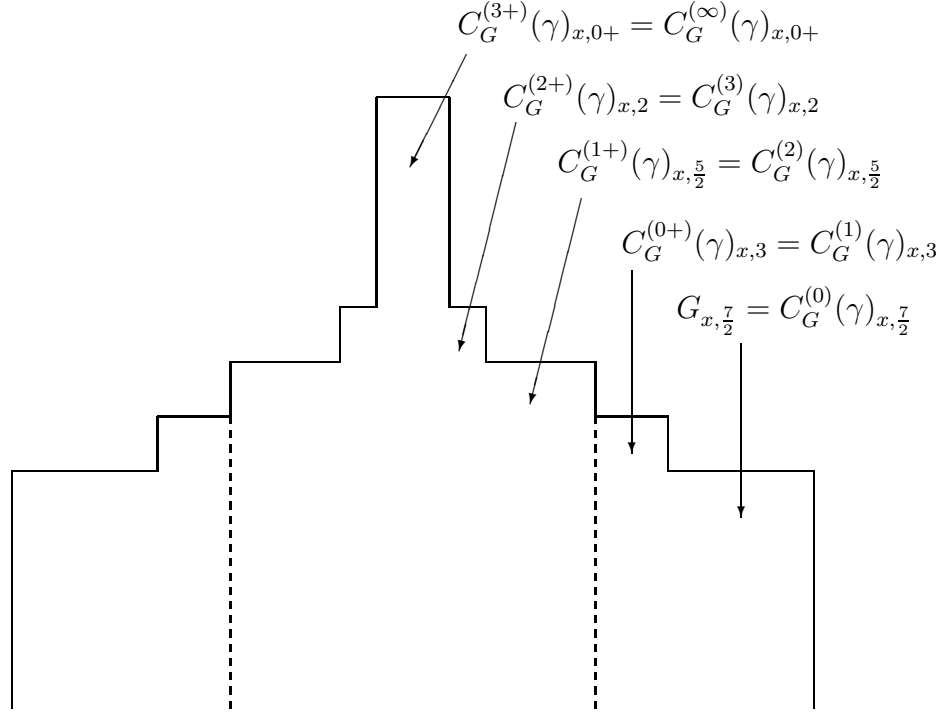
We will also write $\gamma_{<t} = \prod_{0 \le i < t} \gamma_i$ and $\gamma_{\ge t} = \gamma_{<t}^{-1}\gamma$ (so that, with the point $x \in \mathcal{B}(C_{\mathbf{G}}^{(t)}(\gamma), F)$ as above, we have $\gamma_{\ge t} \in C_G^{(r)}(\gamma)_{x,t}$). These elements should be thought of as the "head" and "tail" of γ, respectively. By Corollary 6.14 of *loc. cit.*, $C_{\mathbf{G}}^{(t)}(\gamma) = C_{\mathbf{G}}(\gamma_{<t})^\circ$. Although the head and tail are not independent of the choice of normal t-approximation to γ, they are usually "well determined enough" (as described precisely in Proposition 8.4 of *loc. cit.*) that we need not specify the choice.

If $t > 0$, then we put $\mathcal{B}_t(\gamma) = \left\{ x \in \mathcal{B}(C_{\mathbf{G}}^{(t)}(\gamma), F) \,\middle|\, \mathrm{d}_x(\gamma_{\ge t}) \ge t \right\}$. By Lemma 9.6 of *loc. cit.*, this is uniquely determined, even though $\gamma_{\ge t}$ is not. (An analogous set can also be defined when $t = 0$, as in Definition 9.5 of *loc. cit.*; but we do not need this.) Since $\mathrm{d}_x(\gamma_i) \ge i$ for $0 \le i < t$ and $x \in \mathcal{B}(C_{\mathbf{G}}^{(r)}(\gamma), F)$ (in particular, for $x \in \mathcal{B}_t(\gamma)$), we have that $\mathcal{B}_t(\gamma) \subseteq \mathcal{B}_{t'}(\gamma)$ whenever $t' \in \widetilde{\mathbb{R}}_{>0}$ and $t' \le t$.

If $t \in \widetilde{\mathbb{R}}_{\ge 0}$ and $\gamma \in G$ has a normal t-approximation, then, in the notation of Definition 5.14 of [6], we put $\vec{\mathbf{G}} = (C_{\mathbf{G}}^{(t-i)}(\gamma))_{0 < i \le t}$ and $\vec{s} = (i/2)_{0 < i \le t}$, and write $[\![\gamma; x, t]\!] = \vec{\mathbf{G}}_{x,\vec{s}}$.

We will also need various "truncations" $[\![\gamma; x, t]\!]^{(j)}$ of $[\![\gamma; x, t]\!]$, as in Definition 9.3 of *loc. cit.* These arise by taking only those terms in $\vec{\mathbf{G}}$ and $\vec{s}$ above with $0 < i < 2j$. We will append a subscript G' (writing instead $[\![\gamma; x, t]\!]_{G'}$ or $[\![\gamma; x, t]\!]_{G'}^{(j)}$) to indicate that we are constructing the analogous object, but inside the ambient group G', rather than G.

Definition 1.4.1. If $\vec{\mathbf{G}} = (\mathbf{G}^0, \ldots, \mathbf{G}^d = \mathbf{G})$ is a tame reductive sequence in $\mathbf{G}$, in the sense of Definition 5.1 of [6], and $\vec{r} = (r_0, \ldots, r_d) \in \widetilde{\mathbb{R}}_{\ge 0}^{d+1}$,

$$C_G^{(3+)}(\gamma)_{x,0+} = C_G^{(\infty)}(\gamma)_{x,0+}$$

$$C_G^{(2+)}(\gamma)_{x,2} = C_G^{(3)}(\gamma)_{x,2}$$

$$C_G^{(1+)}(\gamma)_{x,\frac{5}{2}} = C_G^{(2)}(\gamma)_{x,\frac{5}{2}}$$

$$C_G^{(0+)}(\gamma)_{x,3} = C_G^{(1)}(\gamma)_{x,3}$$

$$G_{x,\frac{7}{2}} = C_G^{(0)}(\gamma)_{x,\frac{7}{2}}$$

FIGURE 1. Illustration of $[\![\gamma; x, 7]\!]$ in Example 1.4.2

then we write $\mathcal{T}(\vec{\mathbf{G}}, \vec{r})$ for the set

$$\left\{ \delta \in G \mid \delta \text{ has a normal } r_{d-1}\text{-approximation and } \delta_{<r_i} \in G^i \text{ for } 0 \le i < d \right\}.$$

(Note that r_d is a "dummy number" that has no effect on the resulting set $\mathcal{T}$.)

Example 1.4.2. Suppose that γ has a normal ∞-expansion $(\gamma_i)_{i \ge 0}$ with the property that $\gamma_i = 1$ when $i \notin \mathbb{Z}$ or $i > 3$. Thus, $\gamma = \gamma_0 \gamma_1 \gamma_2 \gamma_3$. Then the group $[\![\gamma; x, 7]\!]$ is a product of various filtration subgroups of centralizer subgroups of G. (See Figure 1.) The larger the centralizer subgroup that is involved, the deeper is the filtration subgroup that appears. The group $[\![\gamma; x, 7]\!]^{(3)}$ corresponds to the region between the vertical dotted lines in Figure 1.

1.5. **Example computation of** $[\![\gamma; x, r]\!]$**.** We give an extended example to illustrate how to compute normal approximations and groups $[\![\gamma; x, r]\!]$ in practice. This involves a considerable amount of notation, all of which should be regarded as being in force for this subsection only.

Suppose that $\mathbf{G} = \mathrm{GL}(V)$ for some finite-dimensional F-vector space V. Let γ be an element of $G = \mathrm{GL}_F(V)$. Suppose for simplicity that γ is compact and semisimple. We describe a recipe for computing the leading

term in a normal approximation to γ. This gives an inductive recipe for computing a normal approximation to γ, from which falls out an explicit description of $C_{\mathbf{G}}^{(r)}(\gamma)$, $\mathcal{B}_r(\gamma)$, and $[\![\gamma; x, r]\!]$ for $r \in \widetilde{\mathbb{R}}_{\geq 0}$. (Although we do not do so here, it is very easy also to compute the group $[\gamma; x, r]$ occurring in Definition 9.3 of [6] from our description.) Remember that we write $[\![\gamma; x, r]\!]_G$ rather than just $[\![\gamma; x, r]\!]$, and $\mathcal{B}_r^{\mathbf{G}}(\gamma)$ rather than just $\mathcal{B}_r(\gamma)$, when we wish to emphasise the ambient group.

Note that, in general, there is no canonical choice of normal approximation. This is reflected in our recipe in the fact that we have to make some choices (namely, of a field L and a uniformizer ϖ_L of L). The point is that, as remarked earlier, the groups $C_{\mathbf{G}}^{(r)}(\gamma)$ and $[\![\gamma; x, r]\!]$, and the set $\mathcal{B}_r(\gamma)$, are nonetheless well defined.

For $\lambda \in (F^{\mathrm{sep}})^{\times}$, write $E_\lambda(V)$ for the minimal γ-stable F-subspace of V such that the action of γ on $V/E_\lambda(V)$ does not have λ as an eigenvalue. Then $E_\lambda(V) = \{0\}$ unless $\lambda \in (F^{\mathrm{sep}})_0^{\times}$; $E_\lambda(V) = E_{\sigma\lambda}(V)$ for $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$; and $V = \bigoplus_{\lambda \in (\dot{F}^{\mathrm{sep}})^{\times}} E_\lambda(V)$, where $(\dot{F}^{\mathrm{sep}})^{\times}$ is a set of representatives for the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ on $(F^{\mathrm{sep}})^{\times}$. We have for $\lambda \in (\dot{F}^{\mathrm{sep}})^{\times}$ that $E_\lambda(V)$ carries the structure of an $F[\lambda]$-vector space, where λ acts by γ.

Put $\mathbf{T} = Z(C_{\mathbf{G}}(\gamma))$, so that $T = \mathbf{T}(F)$ is the set of all $g \in \mathrm{GL}_F(V)$ that act on each $E_\lambda(V)$ as scalar multiplication by an element of $F[\lambda]^{\times}$. Note that $\mathbf{T}$ is maximal if and only if γ is regular. Assume further that γ is *tame*, i.e., that there exists a finite, tamely ramified, Galois extension L of F that contains all of the eigenvalues of γ. Then $\mathbf{T}$ is an L-split, hence tame, torus.

We now choose, for each $d \in \mathbb{R}_{\geq 0}$, a set Λ_d of coset representatives for $L_{d:d+}^{\times}$ as follows. Let ϖ_L be a uniformizer of L such that $\varpi_L^{e(L/F)} \in F$. Write Λ_0 for the set of absolutely semisimple elements of $L_0^{\times}$ (that is, elements whose order is finite and prime to p). If $d > 0$ and $L_d = L_{d+}$, then put $\Lambda_d = \{1\}$. If $d > 0$ and $L_d \neq L_{d+}$, then there is some integer k such that $\varpi_L^k \in L_d \smallsetminus L_{d+}$. Put $\Lambda_d = \{ 1 + \lambda_0 \varpi_d^k \mid \lambda_0 \in \Lambda_0 \}$. It is easy to verify that, for any $d \in \mathbb{R}_{\geq 0}$ and $\lambda \in \Lambda_d$, the stabilizers in $\mathrm{Gal}(L/F)$ of λ and $\lambda L_{d+}^{\times}$ are the same.

Let d be the least index $i \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $E_\lambda(V) \neq 0$ for some $\lambda \in L_i^{\times}$. Then $d = \mathrm{d}(\gamma)$. If $d = \infty$, then $\gamma = 1$; so we assume that $d < \infty$.

Put $\mathbf{G}_\lambda = R_{F[\lambda]/F} \mathrm{GL}(E_\lambda(V))$. Then there is a natural isomorphism of $\prod_{\lambda \in \Lambda_d} \mathbf{G}_\lambda$ with $C_{\mathbf{G}}(\mathbf{T})$, hence a natural injection of it into $\mathbf{G}$. Corresponding to this injection is an injection of $\mathcal{B}(\prod_{\lambda \in \Lambda_d} \mathbf{G}_\lambda, F)$, hence of $\prod_{\lambda \in \Lambda_d} \mathcal{B}(\mathbf{G}_\lambda, F)$, into $\mathcal{B}(\mathbf{G}, F)$ with certain properties (see Proposition 2.1.5 of [42] or Proposition 4.6 of [6] for details). We will regard these injections as inclusions.

Remember that we have chosen a set Λ_d of representatives for $L_{d:d+}^\times$. For $\lambda \in L_d^\times$, write s_λ for the element of $\Lambda_d \cap \lambda L_{d+}^\times$. Let γ_d be the element of $\mathrm{GL}_F(V)$ that acts on $E_\lambda(V)$ by scalar multiplication by $s_\lambda \in F[\lambda]$ for all $\lambda \in L_d^\times$. We claim that $\gamma_d \in T_d$ is good, in the sense of Definition 6.1 of [6]. Indeed, if $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, then we have that $\alpha(\gamma_d) = s_\lambda s_{\lambda'}^{-1}$ for some $\lambda, \lambda' \in L_d^\times$. Suppose that $\alpha(\gamma_d) \in L_{d+}^\times$. Since s_λ and $s_{\lambda'}$ are elements of a set of representatives for $L_{d:d+}^\times$, we have that $s_\lambda = s_{\lambda'}$, hence that $\alpha(\gamma_d) = 1$, as desired.

Further, $\gamma \equiv \gamma_d \pmod{T_{d+}}$. Thus, (γ_d) is a normal $(d+)$-approximation to γ. Put $\gamma_{>d} = \gamma_d^{-1}\gamma \in \mathrm{GL}_F(V)$. By abuse of notation, we will also write $\gamma_{>d}$ for the restriction of this element to any space $E_\lambda(V)$.

The groups $C_{\mathbf{G}}^{(r)}(\gamma)$ and $[\![\gamma; x, r]\!]$, and the sets $\mathcal{B}_r^{\mathbf{G}}(\gamma)$, look different depending on the relative values of r and d. For "small values" of r, we have

$$C_{\mathbf{G}}^{(r)}(\gamma) = \mathbf{G} \qquad\qquad\qquad \text{for } r \le d,$$

and

$$[\![\gamma; x, r]\!]_G = \mathrm{GL}_F(V)_{x,0+} \qquad \text{for } r \le d+ \text{ and } x \in \mathcal{B}(C_{\mathbf{G}}^{(r)}(\gamma), F).$$

For "large values" of r, remember that we have identified $\prod_{\lambda \in \Lambda_d} \mathcal{B}(\mathbf{G}_\lambda, F)$ with a subset of $\mathcal{B}(\mathbf{G}, F)$. If $x \in \mathcal{B}(\mathbf{G}, F)$ lies in this subset and $\lambda \in \Lambda_d$, then we will write x_λ for the image of x under the natural projection to $\mathcal{B}(\mathbf{G}_\lambda, F)$. With this notation, we have

$$C_{\mathbf{G}}^{(r)}(\gamma) = \prod_{\lambda \in \Lambda_d} C_{\mathbf{G}_\lambda}^{(r)}(\gamma_{>d}) \qquad\qquad \text{for } r > d,$$

$$\mathcal{B}_r^{\mathbf{G}}(\gamma) = \prod_{\lambda \in \Lambda_d} \mathcal{B}_r^{\mathbf{G}_\lambda}(\gamma_{>d}) \qquad\qquad \text{for } r \ge d,$$

and

$$[\![\gamma; x, r]\!]_G = G_{x,(r-d)/2} \cdot \prod_{\lambda \in \Lambda_d} [\![\gamma_{>d}; x_\lambda, r]\!]_{G_\lambda} \quad \text{for } r > d+ \text{ and } x \in \mathcal{B}(C_{\mathbf{G}}^{(r)}(\gamma), F).$$

In particular,

$$C_{\mathbf{G}}^{(d+)}(\gamma) = \prod_{\lambda \in \Lambda_d} \mathbf{G}_\lambda$$

and

$$\mathcal{B}_d^{\mathbf{G}}(\gamma) = \prod_{\lambda \in \Lambda_d} \mathcal{B}(\mathbf{G}_\lambda, F).$$

Although the recipe given always works, it can result in normal approximations with more non-1 terms than necessary. In practice, it is usually easy to find a shorter normal approximation to a given element of G. We

illustrate this in case $V = F^3$, so that $G = \mathrm{GL}_3(F)$. There are too many cases to consider them all, so we give only a few representative examples.

If γ is regular elliptic, then we have that T is isomorphic to the multiplicative group of a cubic extension of F. If $\gamma \notin Z(G)T_{d+}$, then (γ) is a normal ∞-approximation to γ. If there exists $r \in \mathbb{R}$ with $r > d$ such that $\gamma \in Z(G)T_r \smallsetminus Z(G)T_{r+}$, say $\gamma = zt$ with $z \in Z(G)$ and $t \in T_r$, then (z, t) is a normal ∞-approximation to γ.

If γ is neither split nor regular elliptic, then we may write $V = V' \oplus V''$, where V' and V'' are γ-stable F-subspaces of V of dimensions 1 and 2 respectively. Write γ' and γ'' for the restrictions of γ to V' and V'', respectively; G'' for $\mathrm{GL}_F(V'')$; and T'' for $Z(C_{G''}(\gamma''))$. When convenient, we will abuse notation and consider an operator on V' to be an operator on V that acts trivially on V'' (and vice versa). Since γ' is an operator on a 1-dimensional vector space, it acts as multiplication by a scalar λ'. We divide this case into subcases depending on which of γ' and γ'' has depth d.

- Suppose that γ' and γ'' both have depth d. If $\gamma'' \notin Z(G'')T''_{d+}$, then (γ) is a normal ∞-approximation to γ. On the other hand, if $\gamma'' \in Z(G'')T''_{d+}$, then $\lambda'^{-1}\gamma'' \in Z(G'')T''_{d+}$. Since γ is not split, there exists $r \in \mathbb{R}$ with $r > d$ such that $\lambda'^{-1}\gamma'' \in Z(G'')T''_r \smallsetminus Z(G'')T''_{r+}$. Write $\lambda'^{-1}\gamma'' = z''t$ with $z'' \in Z(G'')$ and $t \in T''_r$. If z'' has depth d, then $(\lambda' z'', t)$ is a normal ∞-approximation to γ. If $d < \mathrm{d}(z) < r$, then (λ', z'', t) is a normal ∞-approximation to γ.
- Now suppose that γ' has depth d and γ'' has depth $s > d$. Since γ is not split, there exists $r \in \mathbb{R}$ with $r \geq s$ such that $\gamma'' \in Z(G'')T''_r \smallsetminus Z(G'')T''_{r+}$. If $r = s$, then (γ', γ'') is a normal ∞-approximation for γ. If $r > s$, then we may write $\gamma'' = z''t$ with $z'' \in Z(G'')$ and $t \in T''_r \smallsetminus T''_{r+}$, in which case (γ', z'', t) is a normal ∞-approximation for γ.
- The subcase where γ'' has depth d and γ' has depth greater than d is straightforward, but involves several sub-subcases, so we omit it.

There remains the case when γ is split. This is handled similarly, but the plethora of possibilities for depths and congruences among various eigenvalues makes it impractical to give a complete list.

1.6. **Representations and characters.** Recall that, if π is a smooth admissible representation of G, then the character of π is a distribution on G. From work of Harish-Chandra [29] and G. Prasad [4, Appendix B] (see also Corollary A.11 of [12]), this distribution is represented on the regular semisimple set in G by a locally constant function. As mentioned in §0.1, we will denote by Θ_π both the function and the distribution. If H is an open subgroup of G and ρ is a finite-dimensional representation of H, then

we will often denote by θ_ρ the function $h \mapsto \operatorname{tr} \rho(h)$ on H, and by $\dot\theta_\rho$ the function on G that is equal to θ_ρ on H, and to 0 on $G \smallsetminus H$.

2. REVIEW OF J.-K. YU'S CONSTRUCTION

We review here the construction of supercuspidal representations found in [65]. The terminology of this section will remain in force throughout the remainder of the paper.

If ϕ is a character of G and $x \in \mathcal{B}(\mathbf{G}, F)$, then denote by $\mathrm{d}_x(\phi)$ the smallest index $d \in \mathbb{R}_{\geq 0}$ such that ϕ is trivial on $G_{x,d+}$.

Definition 2.1. A *cuspidal datum* is a quintuple $\Sigma = (\vec{\mathbf{G}}, \vec{\phi}, \vec{r}, x, \rho_0')$, where

- $\vec{\mathbf{G}} = (\mathbf{G}^0, \ldots, \mathbf{G}^d = \mathbf{G})$ is a tame Levi F-sequence, and $Z(\mathbf{G}^0)/Z(\mathbf{G})$ is F-anisotropic;
- x lies in $\mathcal{B}(\mathbf{G}^0, F)$, and $\overline{x} \in \mathcal{B}^{\mathrm{red}}(\mathbf{G}^0, F)$ is a vertex;
- $\vec{r} = (r_0, \ldots, r_d)$ is a sequence of real numbers satisfying $0 \leq r_0 < \cdots < r_{d-1} \leq r_d$ and $r_0 > 0$ if $d > 0$;
- $\vec{\phi} = (\phi_0, \ldots, \phi_d)$, where, for each $0 \leq i < d$, ϕ_i is a character of G^i such that $\mathrm{d}_x(\phi_i) = r_i$ and ϕ_i is $\mathbf{G}^{i+1}$-*generic relative to x* (in the sense of [65, §9]), and $\mathrm{d}_x(\phi_d) = r_d$, or $\phi_d = 1$ and $r_d = r_{d-1}$;
- ρ_0' is an irreducible representation of $\mathrm{stab}_{G^0}(\overline{x})$ whose restriction to $G^0_{x,0}$ contains the inflation of a cuspidal representation of $G^0_{x,0:0+}$.

For the remainder of this paper, fix a cuspidal datum Σ, with associated notation as above. For $0 \leq i \leq d$, we have the following objects associated to Σ:

- non-negative real numbers $s_i = r_i/2$;
- subgroups
$$K^i = \mathrm{stab}_{G^0}(\overline{x})(G^0, \ldots, G^i)_{x,(0+,s_0,\ldots,s_{i-1})}$$
and (for $i > 0$)
$$J^i = (G^{i-1}, G^i)_{x,(r_{i-1},s_{i-1})},$$
and
$$J^i_+ = (G^{i-1}, G^i)_{x,(r_{i-1},s_{i-1}+)};$$
- representations ρ_i' of K^i (see Remark 2.2); and
- irreducible supercuspidal representations $\pi_i = \mathrm{Ind}_{K^i}^{G^i}(\rho_i' \otimes \phi_i)$ of depth r_i, in the sense of [46, §3.4].

In particular, $\pi_0 = \mathrm{Ind}_{\mathrm{stab}_{G^0}(\overline{x})}^{G^0} \rho_0' \otimes \phi_0$ is a twist of a depth-zero irreducible supercuspidal representation. Since our calculations offer no new information about depth-zero supercuspidal representations, we assume throughout that $d > 0$.

Remark 2.2. In [65, §11], for $0 \leq i < d$, Yu constructs a canonical representation $\tilde{\phi}_i$ of $\mathrm{stab}_{G^i}(\overline{x}) \ltimes J^{i+1}$. (However, by Proposition 3.26 of [26], one actually has considerable freedom in constructing this representation.) Then, by [65, §4], ρ'_{i+1} is the push-forward of $\tilde{\phi}_i \otimes ((\rho'_i \otimes \phi_i) \ltimes 1)$ along the map $K^i \ltimes J^{i+1} \longrightarrow K^i J^{i+1} = K^{i+1}$.

For $0 \leq i \leq d$, write again ϕ_i for the character of $G^i_{x,s_i+:r_i+}$ induced by ϕ_i. Then there is an element $X_i^* \in \mathfrak{g}^{i*}_{x,-r_i}$ such that

$$\phi_i \circ \mathsf{e}_{x,s_i+:r_i+}\Big|_{\mathfrak{g}^i_{x,s_i+:r_i+}} = \Lambda \circ X_i^*\Big|_{\mathfrak{g}^i_{x,s_i+:r_i+}}.$$

(Note that the right-hand side makes sense as a map on $\mathfrak{g}^i_{x,s_i+:r_i+}$, because $X_i^*(\mathfrak{g}^i_{x,r_i+}) \subseteq F_{0+} \subseteq \ker \Lambda$.) By the definition of genericity, we have $X_i^* \in \mathfrak{z}(\mathfrak{g}^i)^*_{x,-r_i} + \mathfrak{g}^{i*}_{x,(-r_i)+}$. Note that X_i^* is determined only up to translation by $\mathfrak{g}^{i*}_{x,-s_i}$.

For the results of §5.2, we require a hypothesis on the elements X_i^* that is a weaker version of Hypothesis $C(\vec{\mathbf{G}})$ of [26, §2.6]. In particular, by Lemma 2.50 of *loc. cit.*, it holds whenever $\mathbf{G} = \mathrm{GL}_n$.

Hypothesis 2.3. $X_i^* \in \mathfrak{z}(\mathfrak{g}^i)^* + \mathfrak{g}^{i+1*}_{x,-s_i}$ for $0 \leq i < d$.

This hypothesis is used only in the proofs of Corollaries 5.2.6 and 5.2.9 to allow the invocations there of Lemma 4.1. These results, in turn, are necessary only for the computations of Propositions 5.2.12 and 5.2.13. If the hypothesis were dropped, then we could still prove a version of Theorem 7.1, but it would involve the undetermined quantity $\widetilde{\mathfrak{S}}(\phi, \gamma)$ (see §5.2), hence be less explicit.

By Proposition 5.40 of [6], $\mathrm{stab}_{G^i}(\overline{x})G_{x,s_i+} = \mathrm{stab}_{G^i}(\overline{x})(G^i, G)_{x,(r_i+,s_i+)}$. We denote by $\hat{\phi}_i$ the character of $\mathrm{stab}_{G^i}(\overline{x})G_{x,s_i+}$ that agrees with ϕ_i on $\mathrm{stab}_{G^i}(\overline{x})$ and is trivial on $(G^i, G)_{x,(r_i+,s_i+)}$. (In particular, $\hat{\phi}_i$ is trivial on G_{x,r_i+}.) If we write again $\hat{\phi}_i$ for the induced character of $G_{x,s_i+:r_i+}$, then we have that

$$\hat{\phi}_i \circ \mathsf{e}_{x,s_i+:r_i+} = \Lambda \circ X_i^*$$

as maps on $\mathfrak{g}_{x,s_i+:r_i+}$.

In order to study the various π_i via induction in stages, we put $K_{\sigma_i} = \mathrm{stab}_{G^{i-1}}(\overline{x})G^i_{x,0+}$ and $\sigma_i = \mathrm{Ind}_{K^i}^{K_{\sigma_i}} \rho'_i$ for $0 \leq i \leq d$. Since we cannot yet compute the character of the full induced representations π_i in all cases, we will sometimes consider instead the character of $\tau_i := \mathrm{Ind}_{K^i}^{\mathrm{stab}_{G^i}(\overline{x})} \rho'_i \otimes \phi_i = \mathrm{Ind}_{K_{\sigma_i}}^{\mathrm{stab}_{G^i}(\overline{x})} \sigma_i \otimes \phi_i$ for $0 \leq i \leq d$.

Since we will need to use induction on the length d of Σ, we abbreviate $\mathbf{G}' = \mathbf{G}^{d-1}$ (and $G' = \mathbf{G}'(F)$). Further, we put

$$r = r_{d-1}, \ s = s_{d-1}, \ \phi = \phi_{d-1}, \ X^* = X^*_{d-1}, \ \hat{\phi} = \hat{\phi}_{d-1}, \ \tilde{\phi} = \tilde{\phi}_{d-1},$$

$$K = K^d = \operatorname{stab}_{G^0}(\overline{x})\vec{G}_{x,(0+,s_0,\ldots,s_{d-1})},$$

$$K_\sigma = K_{\sigma_d} = \operatorname{stab}_{G'}(\overline{x})G_{x,0+},$$

$$J = J^d = (G',G)_{x,(r,s)},$$

$$J_+ = J^d_+ = (G',G)_{x,(r,s+)},$$

$$\rho' = \rho'_d, \ \sigma = \sigma_d = \operatorname{Ind}_K^{K_\sigma} \rho', \ \tau = \tau_d = \operatorname{Ind}_{K_\sigma}^{\operatorname{stab}_G(\overline{x})}(\sigma \otimes \phi_d),$$

and

$$\pi = \pi_d = \operatorname{Ind}_{K_\sigma}^{G}(\sigma \otimes \phi_d).$$

That is, omitting a sub- or superscript i will be the same as taking $i = d$, except for r, s, ϕ, X^*, $\hat{\phi}$, and $\tilde{\phi}$, where it will be the same as taking $i = d - 1$. The "basic ingredient" in our character formula for π will be $\pi'_0 := \operatorname{Ind}_{K^0}^{G^0} \rho'_0$, a depth-zero supercuspidal representation.

Finally, put $\tilde{\rho} = \operatorname{Ind}_K^{\operatorname{stab}_{G'}(\overline{x})K} \rho'$.

Lemma 2.4. *For $k \in K^{d-1}$ and $j \in J$,*

$$\theta_{\tilde{\rho}}(kj) = \theta_{\tilde{\phi}}(k \ltimes j)\theta_{\tau_{d-1}}(k).$$

Proof. By the Frobenius formula,

$$\theta_{\tilde{\rho}}(kj) = \sum_{g \in K \backslash \operatorname{stab}_{G'}(\overline{x})K} \dot{\theta}_{\rho'}(^g(kj)).$$

We may, and do, actually regard the sum as running over $(K \cap \operatorname{stab}_{G'}(\overline{x})) \backslash \operatorname{stab}_{G'}(\overline{x})$.

Note that, by Lemmata 5.33 and 5.29 of [6],

$$K \cap \operatorname{stab}_{G'}(\overline{x}) = \operatorname{stab}_{G^0}(\overline{x})\vec{G}_{x,\vec{s}} \cap \operatorname{stab}_{G'}(\overline{x})$$

$$= \operatorname{stab}_{G^0}(\overline{x})\vec{G}_{x,\vec{s}(d-1)} = K^{d-1}$$

(where $\vec{s} = (0+, s_0, \ldots, s_{d-2}, s_{d-1})$ and $\vec{s}(d-1) = (0+, s_0, \ldots, s_{d-2}, \infty)$.)

Fix $g \in \operatorname{stab}_{G'}(\overline{x})$. By Corollary 5.21 of [6], $\operatorname{stab}_{G'}(\overline{x})$ normalizes J. In particular, $^gj \in J$. Since $J \subseteq K$, we have that $^g(kj) \in K$ if and only if $^gk \in K$, i.e., if and only if $^gk \in K \cap \operatorname{stab}_{G'}(\overline{x}) = K^{d-1}$. Therefore, either

(1) $^g(kj) \notin K$, so $^gk \notin K^{d-1}$, and

$$\dot{\theta}_{\rho'}(^g(kj)) = 0 = \theta_{\tilde{\phi}}(^gk \ltimes {}^gj)\dot{\theta}_{\rho'_{d-1} \otimes \phi}(^gk);$$

or

(2) $^g(kj) \in K$, so $^gk \in K^{d-1}$, and, by Remark 2.2,

$$\theta_{\rho'}(^g(kj)) = \theta_{\tilde\phi}(^gk \ltimes {}^gj)\theta_{\rho'_{d-1}\otimes\phi}(^gk).$$

Since $\tilde\phi$ is a representation of $\mathrm{stab}_{G'}(\overline{x}) \ltimes J$, its character has the same value at $^gk \ltimes {}^gj = {}^{g\ltimes 1}(k \ltimes j)$ as at $k \ltimes j$ for all $g \in \mathrm{stab}_{G'}(\overline{x})$.

Thus

$$\theta_{\tilde\rho}(kj) = \sum \theta_{\tilde\phi}(^gk \ltimes {}^gj)\dot\theta_{\rho'_{d-1}\otimes\phi}(^gk)$$

$$= \theta_{\tilde\phi}(k \ltimes j) \sum \dot\theta_{\rho'_{d-1}\otimes\phi}(^gk)$$

$$= \theta_{\tilde\phi}(k \ltimes j)\theta_{\tau_{d-1}}(k),$$

where both sums run over those cosets in $K^{d-1}\backslash \mathrm{stab}_{G'}(\overline{x})$ containing an element g such that $^gk \in K^{d-1}$, and the last equality comes from another application of the Frobenius formula and the fact that $\tau_{d-1} = \mathrm{Ind}_{K^{d-1}}^{\mathrm{stab}_{G'}(\overline{x})} \rho'_{d-1}\otimes \phi$. $\qquad\square$

Lemma 2.5. $\mathrm{Res}_{J_+}^{\mathrm{stab}_{G'}(\overline{x})K} \tilde\rho$ and $\mathrm{Res}_{G_{x,r}}^{K_\sigma} \sigma$ are $\hat\phi$-isotypic, and $\mathrm{Res}_{G_{x,r+}}^{\mathrm{stab}_G(\overline{x})} \tau$ is ϕ_d-isotypic.

Proof. The statement about the restriction of $\tilde\rho$ follows from our Lemma 2.4, and Theorem 11.5 of [65] (reproduced as Theorem 3.4 below). For the statement about the restriction of σ, fix $\gamma \in G_{x,r}$, and remember we have the Frobenius formula

$$\theta_\sigma(\gamma) = \sum_{g\in\mathrm{stab}_{G'}(\overline{x})K\backslash K_\sigma} \dot\theta_{\tilde\rho}(^g\gamma).$$

We may, and do, choose coset representatives g in the sum belonging to $G_{x,0+}$, so that $^g\gamma \equiv \gamma \pmod{G_{x,r+}}$. By the first statement, $G_{x,r+} \subseteq \ker\tilde\rho$. Therefore, $\mathrm{Res}_{G_{x,r}}^{K_\sigma} \sigma$ is $\tilde\rho$-isotypic, so the second statement follows from another application of the first statement.

Similarly, for $\gamma \in G_{x,r+}$, we have that $^g\gamma \in G_{x,r+}$, hence that $\theta_\sigma(^g\gamma) = \deg(\sigma)\hat\phi(^g\gamma) = \deg(\sigma)$ (by the second statement), for $g \in \mathrm{stab}_G(\overline{x})$. Thus

$$\theta_\tau(\gamma) = \phi_d(\gamma) \sum_{g\in K_\sigma\backslash \mathrm{stab}_G(\overline{x})} \theta_\sigma(^g\gamma) = [\mathrm{stab}_G(\overline{x}) : K_\sigma]\deg(\sigma)\phi_d(\gamma),$$

and the third statement follows. $\qquad\square$

3. CHARACTERS OF WEIL REPRESENTATIONS

In this section, we make Lemma 2.4 more explicit by computing the character of the representation $\tilde\phi$ appearing there. We begin with two general results on extensions of finite fields. These will be useful in this section, and in §5.2.

In this section, if B is a non-degenerate bilinear or sesquilinear form on a vector space V over a field $\mathbb{F}$, then we will write $\det B$ for the determinant of the matrix of B with respect to some fixed but arbitrary basis of V. Since changing the basis does not change the square class of the resulting determinant, it will not be necessary for our purposes to specify the particular bases chosen.

Lemma 3.1. *Let*

- $\mathbb{E}/\mathbb{F}$ *be a degree-n extension of odd-characteristic finite fields,*
- η_0 *an element of* $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$ *with* $\eta_0^2 = 1$, *and*
- Δ *the determinant of the η_0-Hermitian form on $\mathbb{E}$ given by*

$$(t_1, t_2) \mapsto \mathrm{tr}_{\mathbb{E}/\mathbb{F}}(t_1 \eta_0(t_2)).$$

Then

$$\mathrm{sgn}_{\mathbb{F}}(\Delta) = \left(-\mathrm{sgn}_{\mathbb{F}}(\mathrm{sgn}_{\mathrm{Gal}(\mathbb{E}/\mathbb{F})}(\eta_0)) \right)^{n+1},$$

where $\mathrm{sgn}_{\mathrm{Gal}(\mathbb{E}/\mathbb{F})}$ *is the linear character of* $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$ *whose kernel is the group of squares.*

Proof. Let η be a generator of $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$, and $\left\{ e_i \mid 0 \le i < n \right\}$ a basis for $\mathbb{E}$ over $\mathbb{F}$. The matrix, with respect to the chosen basis, of the indicated pairing is $M \cdot \eta_0(M)^{\mathrm{t}}$, where M is the $n \times n$ matrix with (i,j)th entry $\eta^j e_i$ for $0 \le i, j < n$. Since η induces a permutation of the columns of M that has parity opposite to that of n, we have that $\det M \in \mathbb{F}^{\times}$ (equivalently, $(\det M)^2 \in (\mathbb{F}^{\times})^2$) if and only if n is odd; i.e., $\mathrm{sgn}_{\mathbb{F}}\left((\det M)^2 \right) = (-1)^{n+1}$. Similarly, η_0 induces a permutation of the columns of M that is even or odd according as $\mathrm{sgn}_{\mathrm{Gal}(\mathbb{E}/\mathbb{F})}(\eta_0)^{n+1}$ is 1 or -1, so $\Delta = \det M \cdot \eta_0(\det M) = \mathrm{sgn}_{\mathrm{Gal}(\mathbb{E}/\mathbb{F})}(\eta_0)^{n+1}(\det M)^2$. The result follows. $\square$

There is an analogue of Lemma 3.1 for arbitrary finite cyclic Galois extensions of fields, with essentially the same proof; but its statement is more complicated, so we omit it.

Lemma 3.2. *Let*

- $\mathbb{F}$ *be an odd-characteristic finite field,*
- η_0 *an automorphism of $\mathbb{F}$ with* $\eta_0^2 = 1$,
- $\varepsilon \in \{\pm 1\}$,
- V *a finite-dimensional $\mathbb{F}$-vector space, and*
- B *a non-degenerate (ε, η_0)-Hermitian form on V.*

Then the Witt index of B (see [36, §6.5]) is $\lfloor \dim V/2 \rfloor$ *unless*

- $\dim V$ *is even,*
- $\eta_0 = 1$,
- $\varepsilon = 1$, *and*
- $\det B$ *is not in the square class of* $(-1)^{\dim V/2}$,

in which case it is $(\dim V/2) - 1$.

The Hermitian condition on B means precisely that it is linear in the first variable, and $\varepsilon B(v, w) = \eta_0(B(w, v))$ for $v, w \in V$.

Proof. Let $Q : v \mapsto B(v, v)$ be the quadratic form associated to B, and $\mathbb{F}'$ the fixed field of η_0. Denote by N the map $t \mapsto t \cdot \eta_0(t)$ from $\mathbb{F}$ to $\mathbb{F}'$. Note that the image of N is contained in the set of squares in $\mathbb{F}$.

Let $V = V_+ \oplus V_0 \oplus V_-$ be a Witt decomposition of V, so that V_+ and V_- are maximal totally Q-isotropic subspaces of V that are in duality by B, and $V_0 = (V_+ \oplus V_-)^\perp$ is Q-anisotropic. Then the matrix of B on $V_+ \oplus V_-$, with respect to a suitable basis, is of the form $\begin{pmatrix} 0 & \eta_0(M) \\ \varepsilon M^{\mathrm{t}} & 0 \end{pmatrix}$ for some matrix M; so $\det B\big|_{V_+ \oplus V_-} = (-\varepsilon)^{\dim V_+} \cdot N(\det M)$ belongs to the square class of $(-\varepsilon)^{\dim V_+}$. Let $\mathcal{B}$ be a B-orthogonal basis for V_0, so that $Q\left(\sum_{v \in \mathcal{B}} a_v v\right) = \sum_{v \in \mathcal{B}} N(a_v)Q(v)$ for any constants $a_v \in \mathbb{F}$.

Suppose that $\eta_0 = 1$ and $\varepsilon = 1$. If there are distinct $v_1, v_2 \in \mathcal{B}$ such that $Q(v_2)$ belongs to the square class of $-Q(v_1)$ — say $Q(v_2) = -\lambda^2 Q(v_1)$, with $\lambda \in \mathbb{F}^\times$ — then $Q(\lambda v_1 + v_2) = 0$, which is a contradiction. Thus, if $-1 \in (\mathbb{F}^\times)^2$, then no two elements $Q(v)$, for $v \in \mathcal{B}$, lie in the same square class, implying that $|\mathcal{B}| \leq 2$; and if $-1 \notin (\mathbb{F}^\times)^2$, then all of the elements $Q(v)$, for $v \in \mathcal{B}$, lie in the same square class. In this latter case, if $|\mathcal{B}| > 2$, then let v_1, v_2, and v_3 be distinct elements of $\mathcal{B}$, and write $Q(v_3) = c_i^2 Q(v_i)$, with $c_i \in \mathbb{F}^\times$, for $i = 1, 2$. Then $Q(c_1 \lambda v_1 + c_2 \mu v_2 + v_3) = 0$, where $\lambda, \mu \in \mathbb{F}$ are such that $\lambda^2 + \mu^2 = -1$, which is a contradiction. Thus

- $|\mathcal{B}| \leq 1$, or
- $\mathcal{B} = \{v_1, v_2\}$, and $Q(v_2)$ does not belong to the square class of $-Q(v_1)$.

In the former case, if $\dim V$ is even, then $|\mathcal{B}| = 0$, so $\dim V_+ = \dim V/2$, $V_+ \oplus V_- = V$, and $\det B$ belongs to the square class of $(-1)^{\dim V/2}$. In the latter case, $-\det B\big|_{V_0} = -Q(v_1)Q(v_2)$ is a non-square in $\mathbb{F}$, so $\det B = (\det B\big|_{V_+ \oplus V_-})(\det B\big|_{V_0})$ does not belong to the square class of $(-1)^{\dim V_+}(-1) = (-1)^{\dim V/2}$.

If $\eta_0 = 1$ and $\varepsilon = -1$, then every vector is isotropic for Q, so $V_0 = \{0\}$.

If $\eta_0 \neq 1$, then, since N surjects onto $\mathbb{F}'$, we have that $\left\{ Q(v) \,\middle|\, v \in \mathcal{B} \right\}$ is linearly independent over $\mathbb{F}'$. On the other hand, since B is (ε, η_0)-Hermitian, we have that $\left\{ Q(v) \,\middle|\, v \in \mathcal{B} \right\}$ is contained in the ε-eigenspace for η_0 acting on $\mathbb{F}$, which is 1-dimensional over $\mathbb{F}'$. Thus $|\mathcal{B}| \leq 1$. $\qquad\square$

Theorem 3.3. *(Theorem 4.9.1 of* [25]*.) Let V be an $\mathfrak{f}$-vector space equipped with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$, and ζ an additive character of $\mathfrak{f}$. Denote by W_ζ^V the Weil representation of $\mathrm{Sp}(V)$ associated to ζ (defined in* [25, §2.4]*). Fix $g \in \mathrm{Sp}(V)$.*

(1) If g has no non-zero fixed points, then let V_+ be a maximal g-invariant totally isotropic subspace of V and put $V_0 = V_+^\perp/V_+$. Then

$$\theta_{W_\zeta^V}(g) = \mathrm{sgn}_{\mathfrak{f}}((-1)^{\dim V_0/2} \det(g\big|_{V_+}) \det(g-1\big|_{V_0})).$$

(2) If g fixes pointwise a line $V_+ \subseteq V$, but V_+ does not have a g-invariant complement in $V_+^\perp$, then

$$\theta_{W_\zeta^V}(g) = \theta_{W_\zeta^{V_0}}(g),$$

where $V_0 = V_+^\perp/V_+$.

(3) If g fixes pointwise a line $V_+ \subseteq V$, and V_0 is a g-invariant subspace of $V_+^\perp$ such that $V_+^\perp = V_+ \oplus V_0$, then

$$\theta_{W_\zeta^V}(g) = \theta_{W_\zeta^{V_0}}(g) \sum_{v \in V_0^\perp/V_+} \zeta(\langle gv, v\rangle).$$

In [65, §11], there are described a symplectic structure on J/J_+, an action of $\mathrm{Sp}(J/J_+)$ on $J/\ker\hat\phi$, and an extension of the Weil representation of $\mathrm{Sp}(J/J_+)$ associated to $\hat\phi$ to a representation of $\mathrm{Sp}(J/J_+) \ltimes J/\ker\hat\phi$. We have that $\tilde\phi$ (see Remark 2.2) is the pull-back of this extension to $\mathrm{stab}_{G'}(\overline{x}) \ltimes J$ via the map that restricts to the usual projection $J \longrightarrow J/\ker\hat\phi$, and that takes $k \in \mathrm{stab}_{G'}(\overline{x})$ to the symplectic transformation of J/J_+ induced by the conjugation action of k on J.

Theorem 3.4. *(Theorem 11.5 of [65].) $\tilde\phi\big|_{\{1\}\ltimes J_+}$ is $\hat\phi$-isotypic, and $\tilde\phi\big|_{G'_{x,0+}\ltimes\{1\}}$ is 1-isotypic.*

Proposition 3.5. *The character of $\tilde\phi$ vanishes except on conjugacy classes intersecting $\mathrm{stab}_{G'}(\overline{x}) \ltimes J_+$.*

Proof. This result is proved for an "abstract" Weil representation in [32]. The details of how to apply the result in our situation are in [65, §11]. □

Denote by Γ the Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. For $\eta \in \Gamma$, we will abuse notation and also denote by η the corresponding element of $\mathrm{Gal}(\overline{\mathfrak{f}}/\mathfrak{f})$, where $\overline{\mathfrak{f}}$ is the residue class field of F^{un}.

Fix a bounded-modulo-$Z(G)$ element $\gamma \in G'$ (i.e., an element whose orbits in $\mathcal{B}^{\mathrm{red}}(\mathbf{G}', F)$ are bounded in the sense of metric spaces). By Proposition 2.41 of [62] and Remark 6.9 of [6], γ has a normal $(0+)$-approximation (γ_0) (in G and G'). We assume that $x \in \mathcal{B}_{0+}(\gamma)$. Let $\mathbf{T}$ be a maximal tame F-torus (hence, a tame maximal F-torus) in $\mathbf{G}'$, containing γ_0, such that $x \in \mathcal{B}(\mathbf{T}, F)$; and let E/F be a tame, Galois, strictly Henselian extension over which $\mathbf{T}$ splits.

Notation 3.6. For $\alpha \in \widetilde{\Phi}(\mathbf{G}, \mathbf{T})$, put

- $\Gamma_\alpha = \mathrm{stab}_\Gamma \, \alpha$;
- $F_\alpha = (F^{\mathrm{sep}})^{\Gamma_\alpha}$;
- $e_\alpha = e(F_\alpha/F)$, the ramification index of F_α/F, and $f_\alpha = f(F_\alpha/F)$, the residual degree of F_α/F; and
- $\mathfrak{f}_\alpha = \mathfrak{f}_{F_\alpha}$.

If $-\alpha \in \Gamma \cdot \alpha$, then

- η_α is any element of Γ such that $\eta_\alpha \alpha = -\alpha$;
- $F_{\pm\alpha}$ is the fixed field of $\langle \Gamma_\alpha, \eta_\alpha \rangle$;
- $f_{\pm\alpha} = f(F_{\pm\alpha}/F)$; and
- $\mathfrak{f}_{\pm\alpha} = \mathfrak{f}_{F_{\pm\alpha}}$.

Notation 3.7. In the remainder of §3 only, for $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, denote by $\mathbf{V}_\alpha$ the image of

$$\mathrm{Lie}(\mathbf{G})_\alpha(E) \cap \mathrm{Lie}(\mathbf{G}', \mathbf{G})(E)_{x,(r,s)}$$

in

$$\mathrm{Lie}(\mathbf{G}', \mathbf{G})(E)_{x,(r,s):(r,s+)},$$

and by V_α the set of Γ_α-fixed points in $\mathbf{V}_\alpha$. More concretely, we have that $\mathbf{V}_\alpha = \{0\}$ if $\alpha \in \Phi(\mathbf{G}', \mathbf{T})$; and, if $\alpha \notin \Phi(\mathbf{G}', \mathbf{T})$, then $\mathbf{V}_\alpha \cong {}_E\mathfrak{u}_{(\alpha+s):(\alpha+s)+}$, where $\alpha + s$ is the affine root with gradient α whose value at x is s. Since $\mathbf{T}$ is E-split, we have that $\mathrm{Gal}(F^{\mathrm{sep}}/E)$ acts trivially on $\Phi(\mathbf{G}, \mathbf{T})$, so that $F_\alpha \subseteq E$ — hence, in particular, e_α is not divisible by p — for all $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$. Put $\Xi(\phi) = \{\alpha \in \Phi(\mathbf{G}, \mathbf{T}) \mid V_\alpha \neq \{0\}\}$, and

$$\Xi_1(\phi, \gamma) = \{\alpha \in \Xi(\phi) \mid \alpha(\gamma_0) = 1\} = \Xi(\phi) \cap \Phi(C_{\mathbf{G}}^{(0+)}(\gamma), \mathbf{T}),$$
$$\Xi^1(\phi, \gamma) = \{\alpha \in \Xi(\phi) \mid \alpha(\gamma_0) \neq 1\},$$
$$\Xi_{\mathrm{symm},-1}(\phi, \gamma) = \{\alpha \in \Xi(\phi) \mid -\alpha \in \Gamma \cdot \alpha \text{ and } \alpha(\gamma_0) = -1\} \subseteq \Xi^1(\phi, \gamma),$$
$$\Xi_{\mathrm{symm}}^{-1}(\phi, \gamma) = \{\alpha \in \Xi(\phi) \mid -\alpha \in \Gamma \cdot \alpha \text{ and } \alpha(\gamma_0) \neq \pm 1\} \subseteq \Xi^1(\phi, \gamma),$$

and

$$\Xi^{\mathrm{symm}}(\phi, \gamma) = \{\alpha \in \Xi(\phi) \mid -\alpha \notin \Gamma \cdot \alpha \text{ and } \alpha(\gamma_0) \neq 1\} \subseteq \Xi^1(\phi, \gamma).$$

We will omit ϕ and γ from the notation when convenient. Note that all of these sets are $\Gamma \times \{\pm 1\}$-stable. (Recall that $(-\alpha)(\gamma)$ is $\alpha(\gamma)^{-1}$, *not* $-(\alpha(\gamma))$.) We denote by $\dot{\Xi}_1(\phi, \gamma)$, $\dot{\Xi}_{\mathrm{symm},-1}(\phi, \gamma)$, and $\dot{\Xi}_{\mathrm{symm}}^{-1}(\phi, \gamma)$ sets of representatives for the Γ-orbits in the appropriate sets; and by $\ddot{\Xi}^{\mathrm{symm}}(\phi, \gamma)$ a set of representatives for the $\Gamma \times \{\pm 1\}$-orbits in Ξ^{symm}. Finally, put $\dot{\Xi}_{\mathrm{symm}}(\phi, \gamma) = \dot{\Xi}_{\mathrm{symm},-1} \cup \dot{\Xi}_{\mathrm{symm}}^{-1}$, $\dot{\Xi}^1(\phi, \gamma) = \dot{\Xi}_{\mathrm{symm}} \cup \pm\ddot{\Xi}^{\mathrm{symm}}$, $\dot{\Xi}(\phi, \gamma) = \dot{\Xi}_1 \cup \dot{\Xi}^1$, and $f(\dot{\Xi}_{\mathrm{symm}}(\phi, \gamma)) = \sum_{\alpha \in \dot{\Xi}_{\mathrm{symm}}} f_\alpha$.

Proposition 3.8. *With notation and assumptions as above, we have*

$$\theta_{\tilde{\phi}}(\gamma \ltimes 1) = |(C^{(0+)}_{G'}(\gamma), C^{(0+)}_G(\gamma))_{x,(r,s):(r,s+)}|^{1/2}\, \varepsilon(\phi, \gamma),$$

where

$$\varepsilon(\phi, \gamma) = \mathrm{sgn}_{\mathfrak{f}}(-1)^{f(\dot{\Xi}_{\mathrm{symm}}(\phi,\gamma))/2} \cdot \prod_{\alpha \in \ddot{\Xi}^{\mathrm{symm}}(\phi,\gamma)} \mathrm{sgn}_{\mathfrak{f}_\alpha}(\alpha(\gamma_0))$$
$$\times \prod_{\alpha \in \dot{\Xi}^{-1}_{\mathrm{symm}}(\phi,\gamma)} \mathrm{sgn}_{\mathfrak{f}_\alpha}(1 - \alpha(\gamma_0)).$$

We are using Theorem 3.3 (Theorem 4.9.1 of [25]) in our calculations. Since we are only evaluating the character of the Weil representation at semisimple elements, it may be convenient for some purposes to use Corollary 4.8.1 of [25] to write the sign $\varepsilon(\phi, \gamma)$ in a different form. We do not do this here.

Proof. Write $\mathcal{V} = (G', G)_{x,(r,s):(r,s+)}$, $\mathcal{V}^{(0+)} = (C^{(0+)}_{G'}(\gamma), C^{(0+)}_G(\gamma))_{x,(r,s):(r,s+)}$, $\boldsymbol{\mathcal{V}} = (\mathbf{G'}, \mathbf{G})(E)_{x,(r,s):(r,s+)}$, and $\mathbf{V} = \mathrm{Lie}(\mathbf{G'}, \mathbf{G})(E)_{x,(r,s):(r,s+)}$. Recall that $\mathcal{V} = J/J_+$ carries a symplectic pairing; we will describe it explicitly below. By Corollaries 2.3 and 2.4 of [65], we have an $\mathrm{Int}(\gamma)$-equivariant isomorphism $\boldsymbol{\mathcal{V}} \cong \mathbf{V}$ (essentially, the restriction of $\mathrm{e}^E_{x,s:r}$) that restricts to an isomorphism $j_1 : \mathcal{V} \cong \mathbf{V}^\Gamma$ (also $\mathrm{Int}(\gamma)$-equivariant, of course). We have that $\hat{\phi}([g_1, g_2]) = \Lambda(X^*[j_1(g_1), j_1(g_2)])$ for $g_1, g_2 \in \mathcal{V}$. The $\mathrm{Int}(\gamma)$-equivariant map

$$(X_\alpha)_{\alpha \in \dot{\Xi}} \mapsto \sum_{\alpha \in \dot{\Xi}} \sum_{\eta \in \Gamma/\Gamma_\alpha} \eta(X_\alpha)$$

furnishes an isomorphism $j_2 : \bigoplus_{\alpha \in \dot{\Xi}} V_\alpha \cong \mathbf{V}^\Gamma$ such that $j_1(\mathcal{V}^{(0+)})$ and $j_2(\bigoplus_{\alpha \in \dot{\Xi}^1} V_\alpha)$ are complementary. Thus, there is an $\mathrm{Int}(\gamma)$-equivariant isomorphism

$$(3.1) \qquad \mathcal{V} \cong \mathcal{V}^{(0+)} \oplus \bigoplus_{\alpha \in \dot{\Xi}_{\mathrm{symm}}} V_\alpha \oplus \bigoplus_{\alpha \in \ddot{\Xi}^{\mathrm{symm}}} V_{\pm\alpha} =: V,$$

where $V_{\pm\alpha} = V_\alpha \oplus V_{-\alpha}$ for $\alpha \in \ddot{\Xi}^{\mathrm{symm}}$.

The (additive) pairing on $\mathcal{V}$ is $(g_1, g_2) \mapsto \mathrm{tr}_{\mathfrak{f}/\mathbb{F}_p} X^*[j_1(g_1), j_1(g_2)]$, where $\mathbb{F}_p$ is the finite field with p elements. (In [65, §11], Yu works instead with the multiplicative pairing $(g_1, g_2) \mapsto \hat{\phi}([g_1, g_2])$; but this is identified with our pairing after an appropriate choice of embedding $\mathbb{F}_p \hookrightarrow \mathbb{C}^\times$.) We write B for the pairing on V induced by (3.1). Then $\mathcal{V}^{(0+)}$ is B-orthogonal to V_α for all $\alpha \in \dot{\Xi}$, V_α is B-orthogonal to V_β unless $-\beta \in \Gamma \cdot \alpha$, and
(3.2)
$$B(X, X') = e_\alpha \,\mathrm{tr}_{\mathfrak{f}_\alpha/\mathbb{F}_p} X^*[X, \eta_\alpha X'] \quad \text{for } X, X' \in V_\alpha \text{ with } \alpha \in \dot{\Xi}_{\mathrm{symm}}.$$

In particular, the sums on the right-hand side of (3.1) are B-orthogonal.

Write again γ for the symplectomorphism of V induced by γ. By Corollary 2.5 of [25],

$$(\dagger) \qquad \theta_{\tilde{\phi}}(\gamma \ltimes 1) = \theta_{W_\zeta^{\mathcal{V}^{(0+)}}}(\gamma) \cdot \prod_{\alpha \in \dot{\Xi}_{\text{symm}}} \theta_{W_\zeta^{V_\alpha}}(\gamma) \cdot \prod_{\alpha \in \ddot{\Xi}^{\text{symm}}} \theta_{W_\zeta^{V_{\pm\alpha}}}(\gamma),$$

where ζ is the character induced by the restriction of $\hat{\phi}$ to J_+ (as in [65, §11]).

By Theorem 3.3(3), since γ acts trivially on $\mathcal{V}^{(0+)}$, we have that

$$(3.3) \quad \theta_{W_\zeta^{\mathcal{V}^{(0+)}}}(\gamma) = p^{\dim_{\mathbb{F}_p} \mathcal{V}^{(0+)}/2} = |(C_{G'}^{(0+)}(\gamma), C_G^{(0+)}(\gamma))_{x,(r,s):(r,s+)}|^{1/2}.$$

For $\alpha \in \dot{\Xi}^1$, we have that γ acts on V_α without non-zero fixed points. Thus, our remaining calculations may, and will, use Theorem 3.3(1).

For $\alpha \in \ddot{\Xi}^{\text{symm}}$, we have that V_α is a maximal γ-invariant totally isotropic (indeed, a maximal totally isotropic) subspace of $V_{\pm\alpha}$; and $V_\alpha^\perp/V_\alpha = \{0\}$ (the perpendicular taken in $V_{\pm\alpha}$). Since γ acts on $V_\alpha \cong \mathfrak{f}_\alpha$ by multiplication by $\alpha(\gamma_0)$, we have that $\det_{\mathbb{F}_p}(\gamma|_{V_\alpha}) = N_{\mathfrak{f}_\alpha/\mathbb{F}_p}(\alpha(\gamma_0))$; so, by Theorem 3.3(1),

$$(3.4) \qquad \theta_{W_\zeta^{V_{\pm\alpha}}}(\gamma) = \text{sgn}_{\mathbb{F}_p}(N_{\mathfrak{f}_\alpha/\mathbb{F}_p}(\alpha(\gamma_0))) = \text{sgn}_{\mathfrak{f}_\alpha}(\alpha(\gamma_0)).$$

For $\alpha \in \dot{\Xi}_{\text{symm},-1}$, the restriction of B to V_α is nondegenerate, so $\dim_{\mathbb{F}_p} V_\alpha$ is even. Any maximal totally isotropic subspace of V_α is γ-invariant, so reasoning as above gives

(3.5)
$$\theta_{W_\zeta^{V_\alpha}}(\gamma) = \text{sgn}_{\mathbb{F}_p}(-1)^{\dim_{\mathbb{F}_p} V_\alpha/2} = \text{sgn}_{\mathfrak{f}}(-1)^{\dim_{\mathfrak{f}} V_\alpha/2} = \text{sgn}_{\mathfrak{f}}(-1)^{f_\alpha/2}.$$

(We have used the fact that, if $[\mathfrak{f} : \mathbb{F}_p]$ is even, then $\text{sgn}_{\mathfrak{f}}(-1) = 1$; and, if $[\mathfrak{f} : \mathbb{F}_p]$ is odd, then $\text{sgn}_{\mathfrak{f}}(-1) = \text{sgn}_{\mathbb{F}_p}(-1)$.)

Finally, fix $\alpha \in \dot{\Xi}_{\text{symm}}^{-1}$. Note that $\text{Gal}(F^{\text{sep}}/F_\alpha^{\text{un}})$ acts on $\mathbf{V}_\alpha$ by a linear character, and $\text{Gal}(F_\alpha^{\text{un}}/F_\alpha)$ acts on $\mathbf{V}_\alpha$ via the natural projection to $\text{Gal}(\bar{\mathfrak{f}}/\mathfrak{f}_\alpha)$. Thus, $V_\alpha = \mathbf{V}_\alpha^{\text{Gal}(\bar{\mathfrak{f}}/\mathfrak{f}_\alpha)}$ is a 1-dimensional $\mathfrak{f}_\alpha$-vector space. Choose $X_\alpha \in V_\alpha \setminus \{0\}$, hence an $\mathfrak{f}_\alpha$-linear isomorphism $\iota_\alpha : V_\alpha \longrightarrow \mathfrak{f}_\alpha$, and put $c_\alpha = X^*[X_\alpha, \eta_\alpha X_\alpha] \in \mathfrak{f}_\alpha$, so that $\eta_\alpha c_\alpha = -c_\alpha$. By (3.2), ι_α identifies the restriction of B to V_α with the pairing $(t_1, t_2) \mapsto e_\alpha \text{tr}_{\mathfrak{f}_\alpha/\mathbb{F}_p}(c_\alpha \cdot t_1 \eta_\alpha(t_2))$ on $\mathfrak{f}_\alpha$.

Suppose that W is a γ-invariant totally B-isotropic $\mathbb{F}_p$-subspace of V_α. By abuse of notation, we identify W with its image $\iota_\alpha(W) \subseteq \mathfrak{f}_\alpha$. Since

γ acts on W by multiplication by $\alpha(\gamma_0)$, we have that W is an $\mathbb{F}_p[\alpha(\gamma_0)]$-subspace. Further, for $t_1, t_2 \in W$, we have that

$$e_\alpha \operatorname{tr}_{\mathbb{F}_p[\alpha(\gamma_0)]/\mathbb{F}_p}\left(\lambda \operatorname{tr}_{\mathfrak{f}_\alpha/\mathbb{F}_p[\alpha(\gamma_0)]}(c_\alpha \cdot t_1 \eta_\alpha(t_2))\right)$$
$$= e_\alpha \operatorname{tr}_{\mathfrak{f}_\alpha/\mathbb{F}_p}(c_\alpha \cdot (\lambda t_1)\eta_\alpha(t_2)) = 0$$

for all $\lambda \in \mathfrak{f}[\alpha(\gamma_0)]$; so $B'(t_1, t_2) := \operatorname{tr}_{\mathfrak{f}_\alpha/\mathbb{F}_p[\alpha(\gamma_0)]}(c_\alpha \cdot t_1 \eta_\alpha(t_2)) = 0$ (since e_α is not divisible by p). That is, W is totally B'-isotropic. Conversely, it is clear that an $\mathbb{F}_p[\alpha(\gamma_0)]$-subspace of $\mathfrak{f}_\alpha$ that is totally B'-isotropic is carried by ι_α^{-1} to a γ-invariant totally B-isotropic $\mathbb{F}_p$-subspace of V_α.

Let $\mathfrak{f}_\alpha = (\mathfrak{f}_\alpha)_+ \oplus (\mathfrak{f}_\alpha)_0 \oplus (\mathfrak{f}_\alpha)_-$ be a Witt decomposition for B', so that $(\mathfrak{f}_\alpha)_+$ is a maximal totally B'-isotropic $\mathbb{F}_p[\alpha(\gamma_0)]$-subspace of $\mathfrak{f}_\alpha$. We have that B' is $(-1, \eta_\alpha)$-Hermitian. Denote by $\overline{\alpha(\gamma_0)}$ the image in $\mathfrak{f}_\alpha$ of $\alpha(\gamma_0)$. Since $\eta_\alpha \overline{\alpha(\gamma_0)} = \overline{\alpha(\gamma_0)}^{-1} \neq \overline{\alpha(\gamma_0)}$, we have that η_α is non-trivial on $\mathbb{F}_p[\alpha(\gamma_0)]$. We record two consequences of this fact.

- $\mathbb{F}_p[\alpha(\gamma_0)]$ is not contained in the unique quadratic subfield $\mathfrak{f}_{\pm\alpha} = \mathfrak{f}_\alpha^{\eta_\alpha}$ of $\mathfrak{f}_\alpha$, so $\dim_{\mathbb{F}_p[\alpha(\gamma_0)]} \mathfrak{f}_\alpha$ is odd.
- Put $Q = |\mathbb{F}_p[\alpha(\gamma_0)]^{\eta_\alpha}|$. Then also $\eta_\alpha \overline{\alpha(\gamma_0)} = \overline{\alpha(\gamma_0)}^Q$. This means that $\overline{\alpha(\gamma_0)}^{Q+1} = 1$, so

$$(*) \qquad \operatorname{sgn}_{\mathbb{F}_p[\alpha(\gamma_0)]}(\alpha(\gamma_0)) = \overline{\alpha(\gamma_0)}^{(Q^2-1)/2} = 1.$$

Now Lemma 3.2 gives

$$\dim(\mathfrak{f}_\alpha)_+ = (\dim \mathfrak{f}_\alpha - 1)/2 = \dim(\mathfrak{f}_\alpha)_-,$$

so $\dim(\mathfrak{f}_\alpha)_0 = 1$ (all dimensions being taken over $\mathbb{F}_p[\alpha(\gamma_0)]$). Put $(V_\alpha)_\varepsilon = \iota_\alpha^{-1}\left((\mathfrak{f}_\alpha)_\varepsilon\right)$, where $\varepsilon \in \{+, -, 0\}$. Then $(V_\alpha)_+$ is a maximal γ-stable, totally B-isotropic $\mathbb{F}_p$-subspace of V_α, and $(V_\alpha)_+^\perp/(V_\alpha)_+ \cong (V_\alpha)_0$ (the perpendicular taken in V_α). Since γ acts on $V_\alpha \cong \mathfrak{f}_\alpha$ by multiplication by $\alpha(\gamma_0)$, we have that $\det_{\mathbb{F}_p}(\gamma|_{(V_\alpha)_+}) = N_{\mathbb{F}_p[\alpha(\gamma_0)]/\mathbb{F}_p}(\alpha(\gamma_0))^{(\dim \mathfrak{f}_\alpha - 1)/2}$ and $\det_{\mathbb{F}_p}(\gamma - 1|_{(V_\alpha)_0}) = N_{\mathbb{F}_p[\alpha(\gamma_0)]/\mathbb{F}_p}(\alpha(\gamma_0) - 1)$; so, by Theorem 3.3(1) and $(*)$,

$$\theta_{W_\zeta^{V_\alpha}}(\gamma) = \operatorname{sgn}_{\mathbb{F}_p}(-1)^{\dim_{\mathbb{F}_p} \mathbb{F}_p[\alpha(\gamma_0)]/2} \operatorname{sgn}_{\mathbb{F}_p[\alpha(\gamma_0)]}(\alpha(\gamma_0) - 1)$$

$$(3.6) \qquad\qquad \times \operatorname{sgn}_{\mathbb{F}_p[\alpha(\gamma_0)]}(\alpha(\gamma_0))^{(\dim \mathfrak{f}_\alpha - 1)/2}$$

$$= \operatorname{sgn}_{\mathfrak{f}}(-1)^{f_\alpha/2} \operatorname{sgn}_{\mathfrak{f}_\alpha}(\alpha(\gamma_0) - 1)$$

(where we have used in the last equality the facts that $f_\alpha/2$ is the product of $\dim_{\mathbb{F}_p} \mathbb{F}_p[\alpha(\gamma_0)]/2$ and the odd number $\dim_{\mathbb{F}_p[\alpha(\gamma_0)]} \mathfrak{f}_\alpha$, and that $\operatorname{sgn}_{\mathbb{F}_p[\alpha(\gamma_0)]} = \operatorname{sgn}_{\mathfrak{f}_\alpha}$ on $\mathbb{F}_p[\alpha(\gamma_0)]^\times$).

Upon combining (†) with (3.3)–(3.6), we obtain the desired formula. $\square$

4. VANISHING RESULTS FOR θ_σ

In this section, we consider the support of the character of the representation $\sigma = \sigma_d$ of $K_\sigma = \operatorname{stab}_{G'}(\bar{x})G_{x,0+}$ induced from the representation $\rho' = \rho'_d$ of $K = K^d$. We will show that it is much smaller than would naïvely be expected from an understanding of the support of the character of ρ' (or, better, of $\tilde{\rho}$).

Recall that we have associated to $\phi = \phi_{d-1}$ an element $X^* = X^*_{d-1} \in \mathfrak{z}(\mathfrak{g}')^*_{-r} + \mathfrak{g}'^*_{x,(-r)+}$. For any finite, tamely ramified extension E/F, let $\hat{\phi}_E$ be the linear character of $\mathbf{G}(E)_{x,s+:r+}$ such that $\hat{\phi}_E \circ \mathsf{e}^E_{x,s+:r+} = \Lambda \circ X^*\big|_{\operatorname{Lie}(\mathbf{G})(E)_{x,s+:r+}}$. We will also view $\hat{\phi}_E$ as a character of $\mathbf{G}(E)_{x,s+}$. This is analogous to the definition of $\hat{\phi}$ in §2; in fact, $\hat{\phi}_E\big|_{G_{x,s+}} = \hat{\phi}\big|_{G_{x,s+}}$.

Lemma 4.1. *Suppose that*
- *E/F is a finite, tamely ramified extension,*
- *$d \in \mathbb{R}$ and $d > s$,*
- *$\mathbf{H}$ is a reductive, compatibly filtered E-subgroup of $\mathbf{G}$ (as in Definition 4.3 of [6]),*
- *$\mathbf{H}$ contains an E-split maximal torus $\mathbf{T}$ in $\mathbf{G}'$,*
- *$x \in \mathcal{A}(\mathbf{T})$, and*
- *$X^* \in \operatorname{Lie}(\mathbf{T})^*(E) + \operatorname{Lie}(\mathbf{G}')^*(E)_{x,(-d)+}$.*

Then $\hat{\phi}_E$ is trivial on $(\mathbf{H}, \mathbf{G})(E)_{x,(r+,d)}$.

Proof. We may, and do, assume, after replacing F by E, that $\mathbf{T}$ is F-split and $X^* \in Y^* + \mathfrak{g}'^*_{x,(-d)+}$ with $Y^* \in \mathfrak{t}^*$. Further, there is no harm in taking $d \leq r$. Since $\hat{\phi}_F = \Lambda \circ X^* \circ \mathsf{e}^{-1}_{x,s+:r+}$ as maps on $(H,G)_{x,(r+,d):(r+,r+)}$ (indeed, on $G_{x,s+:r+}$), and since, by Hypothesis A.1(6), $\mathsf{e}_{x,s+:r+}$ carries $\operatorname{Lie}(H,G)_{x,(r+,d):(r+,r+)}$ onto $(H,G)_{x,(r+,d):(r+,r+)}$, it suffices to show that $\Lambda \circ X^*$ is trivial on $\operatorname{Lie}(H,G)_{x,(r+,d)}$.

By an easy analogue of Proposition 5.40 of [6],

$$\operatorname{Lie}(H,G)_{x,(r+,d)} \subseteq \mathfrak{h}_{x,r+} \oplus (\mathfrak{t}^\perp \cap \mathfrak{g}_{x,d}),$$

where $\mathfrak{t}^\perp = \bigoplus_{\alpha \in \Phi(\mathbf{G},\mathbf{T})} \mathfrak{g}_\alpha$. Since $\mathrm{d}_x(X^*) = -r$, we have $\mathfrak{h}_{x,r+} \subseteq \ker(\Lambda \circ X^*)$. For $X \in \mathfrak{t}^\perp \cap \mathfrak{g}_{x,d}$, we have that $X^*(X) \equiv Y^*(X) = 0 \pmod{F_{0+}}$, hence that $\Lambda(X^*(X)) = 1$. $\square$

Remark 4.2. Preserve the notation of Lemma 4.1. Since $\hat{\phi}\big|_{G_{x,s+}} = \hat{\phi}_F$ and $X^* \in \mathfrak{z}(\mathfrak{g}')^* + \mathfrak{g}'^*_{x,(-r)+} \subseteq \mathfrak{t}^* + \mathfrak{g}'^*_{x,(-r)+}$, we always have that $\hat{\phi}$ is trivial on $(H,G)_{x,(r+,r)}$. If $X^* \in \mathfrak{z}(\mathfrak{g}')^* + \mathfrak{g}'^*_{x,-s}$ (for example, if Hypothesis 2.3 is satisfied), then, by applying Lemma 4.1 to a decreasing sequence of d's converging to s, we see that $\hat{\phi}$ is trivial on $(H,G)_{x,(r+,s+)}$.

Proposition 4.3. *Suppose that $t \in \mathbb{R}_{\geq 0}$ and $\gamma \in G$. If*

- *$t < r$,*
- *$t \leq s$, or γ has a normal t-approximation and $x \in \mathcal{B}_t(\gamma)$, and*
- *$\gamma \in \mathrm{stab}_{G'}(\overline{x})G_{x,t} \smallsetminus {}^{G_{x,0+}}(\mathrm{stab}_{G'}(\overline{x})G_{x,t+})$,*

then $\dot{\theta}_\sigma(\gamma) = 0$.

Proof. Note that $\gamma \notin {}^{K_\sigma}(\mathrm{stab}_{G'}(\overline{x})G_{x,t+})$. Recall that $\sigma = \mathrm{Ind}_K^{K_\sigma} \rho'$, so $\mathrm{supp}\,\theta_\sigma \subseteq {}^{K_\sigma} \mathrm{supp}\,\theta_{\rho'}$. By Remark 2.2 and Proposition 3.5, $\mathrm{supp}\,\theta_{\rho'} \subseteq {}^K(\mathrm{stab}_{G'}(\overline{x})G_{x,s+})$. Thus the result is clear for $t \leq s$, so we assume that $t > s$.

By Proposition 9.14 of [6], there is $k \in G_{x,0+}$ such that ${}^k Z_G^{(t)}(\gamma) \subseteq G'$. Since $\theta_\sigma({}^k\gamma) = \theta_\sigma(\gamma)$ and $x = kx \in \mathcal{B}_t({}^k\gamma)$, we may, and do, replace γ by ${}^k\gamma$.

Then, since $x \in \mathcal{B}_t(\gamma)$, we have that $\gamma_{\geq t} \in G_{x,t}$. Now, for $h \in C_G^{(t)}(\gamma)_{x,r-t}$, we have that $[\gamma^{-1}, h] = [\gamma_{\geq t}^{-1}, h] \in G_{x,r}$; so, by Lemma 2.5,

$$\theta_\sigma(\gamma) = \theta_\sigma({}^h\gamma) = \theta_\sigma(\gamma)\hat{\phi}([\gamma_{\geq t}^{-1}, h]) = \theta_\sigma(\gamma) \cdot [\gamma_{\geq t}, \hat{\phi}](h),$$

where $[\gamma_{\geq t}, \hat{\phi}]$ is the character $g \mapsto \hat{\phi}([\gamma_{\geq t}^{-1}, g])$ of $G_{x,r-t}$.

If $[\gamma_{\geq t}, \hat{\phi}]$ is non-trivial on $C_G^{(t)}(\gamma)_{x,r-t}$, then we are done; so suppose that it is trivial there. Then consider $h \in (C_G^{(t)}(\gamma), G)_{x,((r-t)+,r-t)}$. By Lemma 5.32 of [6], $[\gamma_{\geq t}^{-1}, h] \in (C_G^{(t)}(\gamma), G)_{x,(r+,r)}$. By Lemma 4.1 and Remark 4.2, we have that $\hat{\phi}$ is trivial on $(C_G^{(t)}(\gamma), G)_{x,(r+,r)}$. Therefore, $[\gamma_{\geq t}, \hat{\phi}](h) = \hat{\phi}([\gamma_{\geq t}^{-1}, h]) = 1$. Thus $[\gamma_{\geq t}, \hat{\phi}]$ is trivial on $C_G^{(t)}(\gamma)_{x,r-t}$ and $(C_G^{(t)}(\gamma), G)_{x,((r-t)+,r-t)}$, hence, by Proposition 5.40 of *loc. cit.*, on $G_{x,r-t}$. By Lemma B.1, this means that $\gamma_{\geq t} \in (G', G)_{x,(t,t+)}$. Since $\gamma_{<t} \in Z_G^{(r)}(\gamma) \subseteq G'$ and $\gamma_{<t} \in \mathrm{stab}_G(\overline{x})$ (by Remarks 6.10(1) and 6.10(2) of *loc. cit.*, respectively), we have $\gamma = \gamma_{<t}\gamma_{\geq t} \in \mathrm{stab}_{G'}(\overline{x})G_{x,t+}$, which is a contradiction. $\square$

Corollary 4.4. *If γ has a normal r-approximation and $x \in \mathcal{B}_r(\gamma)$, then $\dot{\theta}_\sigma(\gamma) = 0$ unless $\gamma_{<r} \in {}^{G_{x,0+}} \mathrm{stab}_{G'}(\overline{x})$.*

Proof. By Proposition 4.3, $\theta_\sigma(\gamma) = 0$ unless $\gamma \in {}^{G_{x,0+}}(\mathrm{stab}_{G'}(\overline{x})G_{x,r})$. By Corollary 9.16 of [6], $\gamma \in {}^{G_{x,0+}}(\mathrm{stab}_{G'}(\overline{x})G_{x,r})$ if and only if $\gamma_{<r} \in {}^{G_{x,0+}} \mathrm{stab}_{G'}(\overline{x})$. $\square$

Corollary 4.5. *If γ has a normal r-approximation and $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic, then $\dot{\theta}_\sigma(\gamma) = 0$ unless $x \in \mathcal{B}_r(\gamma)$ and $\gamma_{<r} \in {}^{G_{x,0+}} \mathrm{stab}_{G'}(\overline{x})$.*

Proof. This follows from Lemma 9.13 and Corollary 9.16 of [6], and Corollary 4.4. $\square$

Corollary 4.6. *If γ has a normal r-approximation and $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic, then*

$$\dot{\theta}_\sigma(\gamma) = \dot{\theta}_\sigma(\gamma_{<r}) \cdot [G_{x,r}](\gamma_{\geq r})\hat{\phi}(\gamma_{\geq r})$$
$$= \dot{\theta}_\sigma(\gamma_{<r}) \cdot [\mathcal{B}(C_{\mathbf{G}}^{(r)}(\gamma), F)](x) \cdot [G_{x,r}](\gamma_{\geq r})\hat{\phi}(\gamma_{\geq r})$$
$$= [^{G_{x,0+}}G'](\gamma_{<r})\theta_\sigma(\gamma_{<r}) \cdot [\mathcal{B}(C_{\mathbf{G}}^{(r)}(\gamma), F)](x) \cdot [G_{x,r}](\gamma_{\geq r})\hat{\phi}(\gamma_{\geq r}).$$

The notation indicates that $\dot{\theta}_\sigma(\gamma) = \theta_\sigma(\gamma_{<r})\hat{\phi}(\gamma_{\geq r})$ if all the characteristic functions appearing are 1, and $\dot{\theta}_\sigma(\gamma) = 0$ otherwise.

Proof. By Corollary 4.5 $\dot{\theta}_\sigma(\gamma) = 0$ unless $\gamma_{\geq r} \in G_{x,r}$. By Lemma 2.5, if $\gamma_{\geq r} \in G_{x,r}$, then $\dot{\theta}_\sigma(\gamma) = \dot{\theta}_\sigma(\gamma_{<r})\hat{\phi}(\gamma_{\geq r})$. By Corollary 4.5 again, $\dot{\theta}_\sigma(\gamma_{<r}) = 0$ unless $x \in \mathcal{B}(C_{\mathbf{G}}^{(r)}(\gamma), F)$ and $\gamma_{<r} \in {}^{G_{x,0+}}G'$. If $\gamma_{<r} \in {}^{G_{x,0+}}G'$, then $\gamma_{<r}$ is in the domain of σ, so $\dot{\theta}_\sigma(\gamma_{<r}) = \theta_\sigma(\gamma_{<r})$. $\qquad\square$

5. INDUCTION TO $\mathrm{stab}_{G'}(\overline{x})G_{x,0+}$

We have just shown that the character of σ vanishes "far from G'". In this section, we will compute the character on a large subset of $\mathrm{stab}_{G'}(\overline{x})$. By Lemma 2.5, we will then have character values on a large subset of $\mathrm{stab}_{G'}(\overline{x})G_{x,r}$. (To be more precise, unless certain tameness and compactness conditions are satisfied, we must place mild restrictions on the elements at which we evaluate the character. See the following paragraph and the beginning of §6 for details.) The resulting formula (see Proposition 5.3.3) will be expressed in terms of the character of the representation τ_{d-1} of $\mathrm{stab}_{G'}(\overline{x})$ induced from the representation $\rho'_{d-1} \otimes \phi$ of K^{d-1}.

In this section, we suppose that $\gamma \in G'$ has a normal r-approximation $\underline{\gamma} = (\gamma_i)_{0 \leq i < r}$ in G, and that $x \in \mathcal{B}_r(\gamma)$. In particular, by Remark 6.10(2) of [6], we have that $\gamma \in \mathrm{stab}_{G'}(\overline{x})$. We will eventually (after Corollary 5.1.5) also require that γ be semisimple.

5.1. **The Frobenius formula for θ_σ.** The following *ad hoc* definitions are useful for cutting down the number of summands appearing in the Frobenius formula.

Definition 5.1.1. For $g \in G_{x,0+}$, put

$$j(g) = \sup\left\{ j \in \mathbb{R}_{\geq 0} \cup \{\infty\} \mid g \in [\![\gamma; x, r]\!]G_{x,j} \right\}$$

and

$$j^\perp(g) = \sup\left\{ j(g'g) \mid g' \in G'_{x,0+} \right\}.$$

If $j(g) < \infty$, put

$$i(g) = \sup \left\{ i \in \mathbb{R} \cup \{\infty\} \;\middle|\; g \in [\![\gamma; x, r]\!](C_G^{(i)}(\gamma), G)_{x,(j(g),j(g)+)} \right\}$$

and

$$t(g) = \sup \left\{ \mathrm{d}_x([\gamma^{-1}, gh]) \;\middle|\; h \in G_{x,j(g)+} \right\}.$$

If $j^\perp(g) < \infty$, put

$$i^\perp(g) = \sup \left\{ i(g'g) \;\middle|\; g' \in G'_{x,0+}, j(g'g) = j^\perp(g) \right\}$$

and

$$t^\perp(g) = \sup \left\{ t(g'g) \;\middle|\; g' \in G'_{x,0+}, j(g'g) = j^\perp(g) \right\}.$$

The numbers $i(g)$ and $j(g)$ are different measures of how far g is from lying in the group $[\![\gamma; x, r]\!]$. Remember from Figure 1 on page 12 that $[\![\gamma; x, r]\!]$ looks somewhat like a skyscraper that becomes narrower toward the top. The vertical direction represents depth, while horizontal motion toward the center is analogous to moving through successively smaller full-rank reductive subgroups of $\mathbf{G}$ (the connected-centralizer subgroups $C_\mathbf{G}^{(i)}(\gamma)$). The quantity $j(g)$ tells us the vertical distance from g down to a roof of $[\![\gamma; x, r]\!]$. (Of course, if $g \in [\![\gamma; x, r]\!]$, then we have $j(g) = \infty$.) The quantity $i(g)$ answers the question: When we've gone down $j(g)$ floors, landing on a roof of the skyscraper, how far toward the center must we travel in order to hit a wall?

Remark 5.1.2. Since $g, [\gamma^{-1}, g] \in G_{x,0+}$, we have that $j(g), t(g) > 0$. However, it is possible that $i(g) = 0$.

Since $\left\{ j \in \mathbb{R}_{\geq 0} \;\middle|\; G_{x,j} \neq G_{x,j+} \right\}$ is discrete, the supremum in the definition of $j(g)$ is actually a maximum. Suppose that $j(g) < \infty$. Since $\left\{ i \in \mathbb{R} \;\middle|\; C_G^{(i)}(\gamma) \neq C_G^{(i+)}(\gamma) \right\}$ is discrete, the supremum in the definition of $i(g)$ is actually a maximum. If $[\gamma^{-1}, gh] = 1$ for some $h \in G_{x,j(g)+}$, then obviously the supremum in the definition of $t(g)$ is a maximum. Otherwise, $h \mapsto \mathrm{d}_x([\gamma^{-1}, gh])$ is locally constant on the compact set $G_{x,j(g)+}$, so the supremum in the definition of $t(g)$ is again a maximum.

By Proposition 5.40 of [6],

$$[\![\gamma; x, r]\!](C_G^{(r-2j(g))}(\gamma), G)_{x,(j(g),j(g)+)} \subseteq [\![\gamma; x, r]\!]G_{x,j(g)+},$$

so $i(g) < r - 2j(g)$.

If $h \in G_{x,j(g)+}$, then $i(gh) = i(g)$, $j(gh) = j(g)$, and $t(gh) = t(g)$. Since $G_{x,s} \subseteq [\![\gamma; x, r]\!]$, we have that $j(g) < \infty$ if and only if $j(g) < s$, so the functions i, j, and t are all invariant under translation by $G_{x,s}$.

The function $g' \mapsto j(g'g)$ is locally constant on the compact set $G'_{x,0+}$, so the supremum in the definition of $j^\perp(g)$ is actually a maximum. If $j^\perp(g) < \infty$, then $g' \mapsto i(g'g)$ and $g' \mapsto t(g'g)$ are also locally constant on the compact set $\{ g' \in G'_{x,0+} \mid j(g'g) = j^\perp(g) \}$, so the suprema in the definitions of $i^\perp(g)$ and $t^\perp(g)$ are also maxima.

As above, if $j^\perp(g) < \infty$ and $h \in G_{x,j^\perp(g)+}$, then $i^\perp(gh) = i^\perp(g)$, $j^\perp(gh) = j^\perp(g)$, and $t^\perp(gh) = t^\perp(g)$. Furthermore, $j^\perp(g) < \infty$ if and only if $j^\perp(g) < s$, so the functions $i^\perp$, $j^\perp$, and $t^\perp$ are all invariant under translation by $G_{x,s}$. Obviously, they are also invariant under left translation by $G'_{x,0+}$, hence (by Proposition 5.40 of [6]) also by $(G', G)_{x,(0+,s)}$ and (if $j^\perp(g) < \infty$) by $(G', G)_{x,(0+,j^\perp(g)+)}$.

Lemma 5.1.3. *Fix $g \in G_{x,0+}$. If $j(g) < \infty$, then $t(g) = i(g) + j(g)$.*

Proof. Put $i_0 = i(g)$, $j_0 = j(g)$, and $t_0 = t(g)$, so $g \in [\![\gamma; x, r]\!](C_G^{(i_0)}(\gamma), G)_{x,(j_0,j_0+)}$.

Since $i_0 < r - 2j_0$, we have by Proposition 5.40 and Remark 6.7(4) of [6] that

$$[\![\gamma; x, r]\!](C_G^{(i_0)}(\gamma), G)_{x,(j_0,j_0+)} = [\![\gamma; x, r]\!]^{(j_0)} C_G^{(i_0)}(\gamma)_{x,j_0} G_{x,j_0+}.$$

Choose $h \in [\![\gamma; x, r]\!]^{(j_0)} g G_{x,j_0+} \cap C_G^{(i_0)}(\gamma)_{x,j_0}$. Then there is $k_1 \in G_{x,j_0+}$ such that $h \in [\![\gamma; x, r]\!]^{(j_0)} g k_1$.

Note that $[\gamma^{-1}, h] = [\gamma_{\geq i_0}^{-1}, h] \in G_{x,i_0+j_0}$. If $[\gamma^{-1}, h] \in G_{x,(i_0+j_0)+}$, then Lemma 9.8 of *loc. cit.* gives $h \in (C_G^{(i_0+)}(\gamma), G)_{x,(j_0,j_0+)}$, so $g \in [\![\gamma; x, r]\!](C_G^{(i_0+)}(\gamma), G)_{x,(j_0,j_0+)}$, contradicting the definition of i_0. Thus $\mathrm{d}_x([\gamma^{-1}, h]) = i_0 + j_0$.

Suppose that $[\gamma^{-1}, h] \in G_{x,t_0+}$. Then, by Remarks 6.7(1) and 6.10(3) of *loc. cit.*, the fact that $g k_1 h^{-1} \in [\![\gamma; x, r]\!]^{(j_0)}$ implies that $[\gamma^{-1}, g k_1 h^{-1}] \in G_{x,r-j_0}$, hence that $[\gamma^{-1}, h] \equiv [\gamma^{-1}, g k_1] \pmod{G_{x,r-j_0}}$. By the definition of $t(g)$, we have $[\gamma^{-1}, g k_1] \notin G_{x,t_0+}$. Thus $r - j_0 \leq t_0$, so $[\gamma^{-1}, h] \in G_{x,r-j_0}$. By Lemma 9.8 of *loc. cit.*, $h \in (C_G^{(r-2j_0)}(\gamma), G)_{x,(j_0,j_0+)} \subseteq (C_G^{(i_0+)}(\gamma), G)_{x,(j_0,j_0+)}$, so $g \in [\![\gamma; x, r]\!](C_G^{(i_0+)}(\gamma), G)_{x,(j_0,j_0+)}$, contradicting the definition of $i(g)$.

Thus $[\gamma^{-1}, h] \notin G_{x,t_0+}$, so $i_0 + j_0 \leq t_0$. Suppose that $i_0 + j_0 < t_0$. Then there is $k_2 \in G_{x,j_0+}$ such that $[\gamma^{-1}, g k_2] \in G_{x,t_0} \subseteq G_{x,(i_0+j_0)+}$. We have $h \in [\![\gamma; x, r]\!]^{(j_0)} G_{x,j_0+} g k_2$ — say, $h = h' g k_2$, with $h' \in [\![\gamma; x, r]\!]^{(j_0)} G_{x,j_0+}$. Since $[\![\gamma; x, r]\!]^{(j_0)} \subseteq [\![\gamma_{\geq i_0}; x, r]\!]^{(j_0)} \subseteq [\gamma_{\geq i_0}; x, (i_0 + j_0)+]$, and clearly $G_{x,j_0+} \subseteq [\gamma_{\geq i_0}; x, (i_0 + j_0)+]$, we have that

$$[\gamma^{-1}, h] = [\gamma_{\geq i_0}^{-1}, h] = [\gamma_{\geq i_0}^{-1}, h'] \cdot {}^{h'}[\gamma_{\geq i_0}^{-1}, g k_2] \in G_{x,(i_0+j_0)+} \cdot {}^{h'}[\gamma_{\geq i_0}^{-1}, g k_2].$$

By Lemma 9.8 of [6], $g k_2 \in [\gamma; x, t_0]$, where $[\gamma; x, t_0]$ is as in Definition 9.3 of *loc. cit.* Since $[\gamma; x, t_0] \subseteq [\gamma_{\geq i_0}; x, t_0]$, we have by Remark 6.10(3) of *loc.*

cit. that $[\gamma_{\geq i_0}^{-1}, gk_2] \in G_{x,t_0} \subseteq G_{x,(i_0+j_0)+}$, so also $[\gamma^{-1}, h] \in G_{x,(i_0+j_0)+}$, a contradiction. $\qquad\square$

Corollary 5.1.4. *Fix* $g \in G_{x,0+}$*. If* $j^\perp(g) < \infty$*, then* $t^\perp(g) = i^\perp(g) + j^\perp(g)$*.*

Proof. By Remark 5.1.2, there is some $g' \in G'_{x,0+}$ such that $i(g'g) = i^\perp(g)$ and $j(g'g) = j^\perp(g)$. Thus $t^\perp(g) \geq t(g'g) = i(g'g) + j(g'g) = i^\perp(g) + j^\perp(g)$. Similarly, there is some $g'' \in G'_{x,0+}$ such that $j(g''g) = j^\perp(g)$ and $t(g''g) = t^\perp(g)$. Thus $t^\perp(g) = t(g''g) = i(g''g) + j(g''g) \leq i^\perp(g) + j^\perp(g)$. $\qquad\square$

Corollary 5.1.5. *Fix* $g \in G_{x,0+}$*. If* $j(g) < \infty$*, then* $t(g) < r - j(g)$*. If* $j^\perp(g) < \infty$*, then* $t^\perp(g) < r - j^\perp(g)$*.*

From now on, suppose that γ is semisimple. This condition can be removed if desired by observing that all the functions in which we will be interested are locally constant; but, since we will only use the main result of this section (Proposition 5.3.3) when the semisimplicity condition is already satisfied, it is not a serious restriction.

By Lemma 8.2 of [6], $\underline{\gamma}$ is a normal r-approximation to γ in G', so that it makes sense to speak of groups such as $C_{G'}^{(i^\perp(g))}(\gamma)$ below.

Lemma 5.1.6. *Fix* $g \in G_{x,0+}$ *with* $j^\perp(g) < \infty$*. There is* $g^\perp \in (G', G)_{x,(0+,j^\perp(g)+)} \cdot g$ *such that*

$$[\gamma^{-1}, g^\perp] \in (C_{G'}^{(i^\perp(g))}(\gamma), C_G^{(i^\perp(g))}(\gamma))_{x,(t^\perp(g)+,t^\perp(g))}.$$

Proof. Put $i_0 = i^\perp(g)$, $j_0 = j^\perp(g)$, and $t_0 = t^\perp(g)$, so $i_0 + j_0 = t_0$ (by Corollary 5.1.4) and $t_0 < r - j_0$ (by Corollary 5.1.5). Put also $\mathbf{H} = C_{\mathbf{G}}^{(i_0)}(\gamma)$ and $\mathbf{H}' = C_{\mathbf{G}'}^{(i_0)}(\gamma)$. By Remark 5.1.2, there is $g' \in G'_{x,0+}$ such that $i(g'g) = i_0$ and $j(g'g) = j_0$. By Lemma 5.1.3, we have that $t(g'g) = i(g'g) + j(g'g) = i_0 + j_0 = t_0$. In particular, $g'g \in [\![\gamma; x, r]\!](H, G)_{x,(j_0,j_0+)}$. By Remark 6.7(4) and Proposition 5.40 of [6],

$$[\![\gamma; x, r]\!](H, G)_{x,(j_0,j_0+)} = [\![\gamma; x, r]\!]^{(j_0)}(H, G)_{x,(j_0,j_0+)}$$

and

$$(H, G)_{x,(j_0,j_0+)} = (H', G)_{x,(j_0,j_0+)}(H', H)_{x,(j_0+,j_0)}.$$

Since the commutator of $G_{x,0+}$ with $(H, G)_{x,(j_0,j_0+)} \subseteq G_{x,j_0}$ lies in $G_{x,j_0+} \subseteq (H, G)_{x,(j_0,j_0+)}$, we have that $[\![\gamma; x, r]\!]^{(j_0)} \subseteq G_{x,0+}$ normalizes $(H, G)_{x,(j_0,j_0+)}$. Thus we may write $g'g = k'g_1k_-$, with $k' \in (H', G)_{x,(j_0,j_0+)}$, $g_1 \in (H', H)_{x,(j_0+,j_0)}$, and $k_- \in [\![\gamma; x, r]\!]^{(j_0)}$. Since $\gamma_{\geq i_0} \in H'_{x,i_0}$ (or $\mathrm{stab}_{H'}(\overline{x})$, if $i_0 = 0$), Lemma

5.32 (or Corollary 5.21, if $i_0 = 0$) of [6] gives

$$[\gamma^{-1}, g_1] = [\gamma_{\geq i_0}^{-1}, g_1] \in (H', H)_{x,(t_0+,t_0)}.$$

Further, Remarks 6.7(1) and 6.10(3) of *loc. cit.* give

$$[\gamma^{-1}, k_-] \in H_{x,r-j_0} \subseteq H_{x,t_0+},$$

hence ${}^{g_1}[\gamma^{-1}, k_-] \in H_{x,t_0+}$. Thus, if $g^{\perp} = g_1 k_-$, then

$$[\gamma^{-1}, g^{\perp}] = [\gamma^{-1}, g_1] \cdot {}^{g_1}[\gamma^{-1}, k_-] \in (H', H)_{x,(t_0+,t_0)}.$$

Since $g^{\perp} g^{-1} = k'^{-1} g' \in (G', G)_{x,(0+,j_0+)}$, we are done. $\qquad\square$

5.2. **Gauss sums.**
In this section, we consider (in the context of our character computations) certain sums associated to non-degenerate quadratic forms on vector spaces over finite fields. We call these sums Gauss sums since, for a 1-dimensional vector space, they are a special case of classical Gauss sums (see [43, §5.2]). We begin with a simple result that computes such objects.

Definition 5.2.1. Recall that, in §1.1, we chose a square root $\sqrt{-1}$ of -1 and used it to construct a non-trivial additive character Λ of $\mathfrak{f}$. Put

$$\mathfrak{G}_{\Lambda}(\mathfrak{f}) = \begin{cases} -(-1)^{\log_p |\mathfrak{f}|}, & p \equiv 1 \pmod 4 \\ (-\sqrt{-1})^{\log_p |\mathfrak{f}|}, & p \equiv 3 \pmod 4. \end{cases}$$

Lemma 5.2.2. *Let V be a finite-dimensional $\mathfrak{f}$-vector space, and B a non-degenerate symmetric bilinear form on V. Put $\mathfrak{G}(V, B) = |V|^{-1/2} \sum_{v \in V} \Lambda(B(v, v))$. Then*

$$\mathfrak{G}(V, B) = \mathrm{sgn}_{\mathfrak{f}}(\det B) \mathfrak{G}_{\Lambda}(\mathfrak{f})^{\dim_{\mathfrak{f}} V}.$$

Proof. Notice that, if V is 1-dimensional and $v_0 \in V \smallsetminus \{0\}$, then

$$\sum_{v \in V} \Lambda(B(v, v)) = \sum_{t \in \mathfrak{f}} \Lambda(t^2 B(v_0, v_0))$$

$$= \sum_{t \in \mathfrak{f}} \mathrm{sgn}_{\mathfrak{f}}(t) \Lambda(t B(v_0, v_0)) = \mathrm{sgn}_{\mathfrak{f}}(B(v_0, v_0)) \sum_{t \in \mathfrak{f}} \mathrm{sgn}_{\mathfrak{f}}(t) \Lambda(t),$$

where $\mathrm{sgn}_{\mathfrak{f}}(0) = 1$. By Theorem 5.15 of [43], this latter sum equals $\mathrm{sgn}_{\mathfrak{f}}(B(v_0, v_0)) \mathfrak{G}_{\Lambda}(\mathfrak{f}) = \mathrm{sgn}_{\mathfrak{f}}(\det B) \mathfrak{G}_{\Lambda}(\mathfrak{f})^{\dim V}$. Now notice that, if $V = \bigoplus_{i \in I} V_i$ is an orthogonal direct sum decomposition and, for $i \in I$, B_i denotes the restriction of B to $V_i \times V_i$, then we have (with the obvious notation) $\mathfrak{G}(V, B) = \prod_{i \in I} \mathfrak{G}(V_i, B_i)$. By Theorem 6.21 of [43], we are done. $\qquad\square$

Remark 5.2.3. It is also possible to compute a Gauss sum as in Lemma 5.2.2 by re-writing it as $\sum_{b \in \mathfrak{f}} N_b \Lambda(b)$, where $N_b = \left| \left\{ v \in V \mid B(v, v) = b \right\} \right|$ for all $b \in \mathfrak{f}$. We can then use the explicit computations of N_b in Theorems 6.26

and 6.27 of [43], together with the fact that $\mathfrak{G}_\Lambda(\mathfrak{f})^2 = \mathrm{sgn}_\mathfrak{f}(-1)$, to obtain the desired result.

Definition 5.2.4. Put

$$\widetilde{\mathfrak{G}}(\phi, \gamma) = \sum_{g \in [\![\gamma;x,r]\!]_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \setminus [\![\gamma;x,r]\!]^{(s)}} \hat{\phi}([\gamma^{-1}, g])$$

and

$$\mathfrak{G}(\phi, \gamma) = |\widetilde{\mathfrak{G}}(\phi, \gamma)|^{-1} \widetilde{\mathfrak{G}}(\phi, \gamma).$$

In this subsection, we will compute $|\widetilde{\mathfrak{G}}| = |\widetilde{\mathfrak{G}}(\phi, \gamma)|$ and $\mathfrak{G} = \mathfrak{G}(\phi, \gamma)$. The proof of the main result (Proposition 5.2.13) is quite close, in structure and content, to that of Proposition 3.8. A similar result appears in [64, §VIII.5].

Recall (see Definition 6.8 of [6]) that, since γ is a normal r-approximation to γ in $\mathbf{G}'$, in particular $\gamma_{<r}$ is tame in $\mathbf{G}'$. Let $\mathbf{T}$ be a maximal F-tame (hence, since $\mathbf{G}'$ is F-tame, an F-tame maximal) torus in $\mathbf{G}'$ containing $\gamma_{<r}$. Let E/F be the splitting field of $\mathbf{T}$. Recall that we defined, at the beginning of §5, a character $\hat{\phi}_E$ of $\mathbf{G}(E)_{x,s+}$ that extends $\hat{\phi}|_{G_{x,s+}}$ and is trivial on $\mathbf{G}(E)_{x,r+}$.

For $g, g_1, g_2 \in [\![\gamma; x, r]\!]_{\mathbf{G}(E)}^{(s)}$, put

$$\mathcal{Q}(g) = \hat{\phi}_E([\gamma^{-1}, g]) \quad \text{and} \quad \mathcal{B}(g_1, g_2) = \hat{\phi}_E([[g_2, \gamma^{-1}], g_1]).$$

Except in Corollary 5.2.9, we will be interested only in the restrictions of $\mathcal{Q}$ and $\mathcal{B}$ to $[\![\gamma; x, r]\!]^{(s)}$ (respectively, $[\![\gamma; x, r]\!]^{(s)} \times [\![\gamma; x, r]\!]^{(s)}$); however, we could not find a proof of Corollary 5.2.9 that did not involve passing to extension fields. We will show that $\mathcal{Q}$ is, in some sense, a quadratic form (see Corollary 5.2.9), so that we can realize $\mathfrak{G}$ as a Gauss sum (see Proposition 5.2.13).

It is straightforward to verify that

$$(5.2.1) \qquad \mathcal{Q}(g_1 g_2) = \mathcal{Q}(g_1)\mathcal{Q}(g_2)\mathcal{B}(g_1, g_2) \quad \text{for } g_1, g_2 \in [\![\gamma; x, r]\!]_{\mathbf{G}(E)}^{(s)}.$$

By Proposition 5.40 of [6], any element $h \in [\![\gamma; x, r]\!]_{\mathbf{G}(E)}^{(s)}$ may be written as

$$(5.2.2) \qquad h = \prod_{0 < i < r} h_i, \quad \text{with } h_i \in C_{\mathbf{G}}^{(i)}(\gamma)(E)_{x,(r-i)/2} \text{ for } 0 < i < r.$$

If $g_1, g_2 \in [\![\gamma; x, r]\!]_{\mathbf{G}(E)}^{(s)}$ and $g_j = \prod_{0<i<r} g_{ji}$ are decompositions as in (5.2.2) for $j = 1, 2$, then one verifies inductively that there are elements $g_i' \in$

$g_{1i} g_{2i} C_{\mathbf{G}}^{(i)}(\gamma)(E)_{x,((r-i)/2)+}$ for $0 < i < r$ such that $\left(\prod_{i_0 < i < r} g_{1i} \right) \left(\prod_{i_0 < i < r} g_{2i} \right) = \prod_{i_0 < i < r} g_i'$ for $0 \leq i_0 < r$. In particular, for $i_0 = 0$, we obtain

(5.2.3) $$g_1 g_2 = \prod_{0 < i < r} g_i'.$$

Lemma 5.2.5. *If $g \in [\![\gamma; x, r]\!]_{\mathbf{G}(E)}^{(s)}$ and $g = \prod_{0 < i < r} g_i$ is a decomposition as in (5.2.2), then $\mathcal{Q}(g) = \prod_{0 < i < r} \mathcal{Q}(g_i)$.*

Proof. One verifies inductively that $[\gamma^{-1}, g] \equiv \prod_{i_0 < i < r} [\gamma^{-1}, g_i] \pmod{\mathbf{G}(E)_{x,r+}}$ for $0 \leq i_0 < r$. Since $\mathbf{G}(E)_{x,r+} \subseteq \ker \hat{\phi}_E$, evaluating $\hat{\phi}_E$ at both sides of the above identity for $i_0 = 0$ gives the desired result. $\square$

Corollary 5.2.6. *$\mathcal{Q}$ is constant on cosets of $[\![\gamma_{<r}; x, r+]\!]_{\mathbf{G}(E)}^{(s)}$. The restriction of $\mathcal{Q}$ to $[\![\gamma; x, r]\!]^{(s)}$ is constant on right cosets of $[\![\gamma; x, r]\!]_{G'}^{(s)}$.*

The appearance of $[\![\gamma_{<r}; x, r+]\!]_{\mathbf{G}(E)}^{(s)}$ in the statement of the corollary is somewhat unexpected. It appears because γ itself might not have a normal $(r+)$-approximation. If (as will usually be the case, by Lemma 8.1 of [6]) it does have such an approximation, then $[\![\gamma; x, r+]\!]_{\mathbf{G}(E)}^{(s)} = [\![\gamma_{<r}; x, r+]\!]_{\mathbf{G}(E)}^{(s)}$.

Proof. By Proposition 5.40 of [6], any element $g_+ \in [\![\gamma_{<r}; x, r+]\!]_{\mathbf{G}(E)}^{(s)}$ may be written as $g_+ = \prod_{0 < i < r} g_{+,i}$, with $g_{+,i} \in C_{\mathbf{G}}^{(i)}(\gamma_{<r})(E)_{x,((r-i)/2)+} = C_{\mathbf{G}}^{(i)}(\gamma)(E)_{x,((r-i)/2)+}$ for $0 < i < r$. If $g \in [\![\gamma; x, r]\!]_{\mathbf{G}(E)}^{(s)}$ and $g = \prod_{0 < i < r} g_i$ is a decomposition as in (5.2.2), then, by (5.2.3) and Lemma 5.2.5, $\mathcal{Q}(g) = \prod_{0 < i < r} \mathcal{Q}(g_i)$ and $\mathcal{Q}(g_+ g) = \prod_{0 < i < r} \mathcal{Q}(g_i')$, where $g_i' \in g_{+,i} g_i C_{\mathbf{G}}^{(i)}(\gamma)(E)_{x,((r-i)/2)+} = C_{\mathbf{G}}^{(i)}(\gamma)(E)_{x,((r-i)/2)+} g_i$ for $0 < i < r$, so it suffices to show that $\mathcal{Q}(g_i') = \mathcal{Q}(g_i)$ for $0 < i < r$. Indeed, for such i, put $k_i = g_i' g_i^{-1} \in C_{\mathbf{G}}^{(i)}(\gamma)(E)_{x,((r-i)/2)+}$. Upon applying Lemma 5.2.5 again (this time, with k_i playing the role of $g_{i-\varepsilon}$ for ε sufficiently small), we obtain that $\mathcal{Q}(g_i') = \mathcal{Q}(k_i)\mathcal{Q}(g_i) = \hat{\phi}_E([\gamma^{-1}, k_i])\mathcal{Q}(g_i)$. Further,

$$[\gamma^{-1}, k_i] \equiv [\gamma_{<r}^{-1}, k_i] \pmod{\mathbf{G}(E)_{x,r+} \subseteq \ker \hat{\phi}_E},$$

so it suffices to show that $[\gamma_{<r}^{-1}, k_i] \in \ker \hat{\phi}_E$. Since

$$[\gamma_{<r}^{-1}, k_i] \in \mathbf{G}(E)_{x,((r+i)/2)+} = (C_{\mathbf{G}}^{(r)}(\gamma), \mathbf{G})(E)_{x,(((r+i)/2)+,((r+i)/2)+)}$$

and $\gamma_{<r} \in Z(C_G^{(r)}(\gamma))$, Lemma 5.30 of [6] gives

$$[\gamma_{<r}^{-1}, k_i] \in (C_{\mathbf{G}}^{(r)}(\gamma), \mathbf{G})(E)_{x,(r+,((r+i)/2)+)},$$

and Lemma 4.1, Remark 4.2, and Hypothesis 2.3 give that $[\gamma_{<r}^{-1}, k_i] \in \ker \hat{\phi}_E$, as desired.

If $g \in [\![\gamma; x, r]\!]^{(s)}$ and $g' \in [\![\gamma; x, r]\!]_{G'}^{(s)}$, then, since $\hat{\phi}_E\big|_{G_{x,s+}} = \hat{\phi}\big|_{G_{x,s+}}$ and $\hat{\phi}$ is invariant under conjugation by $G'_{x,0+}$, we have

$$
\begin{aligned}
\mathcal{Q}(g'g) = \hat{\phi}_E([\gamma^{-1}, g'g]) &= \hat{\phi}([\gamma^{-1}, g'g]) \\
&= \hat{\phi}(\gamma^{-1})\hat{\phi}^{g'}(\gamma)\hat{\phi}^{g'}([\gamma^{-1}, g]) = \hat{\phi}([\gamma^{-1}, g]) \\
&= \hat{\phi}_E([\gamma^{-1}, g]) = \mathcal{Q}(g). \quad \square
\end{aligned}
$$

Lemma 5.2.7. *If* $g_1, g_2 \in [\![\gamma; x, r]\!]_{\mathbf{G}(E)}^{(s)}$ *and* $g_j = \prod_{0<i<r} g_{ji}$ *are decompositions as in* (5.2.2) *for* $j = 1, 2$, *then* $\mathcal{B}(g_1, g_2) = \prod_{0<i<r} \mathcal{B}(g_{1i}, g_{2i})$.

Proof. By (5.2.1), (5.2.3), Lemma 5.2.5, and Corollary 5.2.6, we have that, for suitable g'_i,

$$
\begin{aligned}
\mathcal{Q}(g_1)\mathcal{Q}(g_2)\mathcal{B}(g_1, g_2) = \mathcal{Q}(g_1 g_2) &= \prod_{0<i<r} \mathcal{Q}(g'_i) = \prod_{0<i<r} \mathcal{Q}(g_{1i} g_{2i}) \\
&= \prod_{0<i<r} \mathcal{Q}(g_{1i})\mathcal{Q}(g_{2i})\mathcal{B}(g_{1i}, g_{2i}) = \mathcal{Q}(g_1)\mathcal{Q}(g_2) \prod_{0<i<r} \mathcal{B}(g_{1i}, g_{2i}). \quad \square
\end{aligned}
$$

For $0 < i < r$, let Y_i be any element of $\mathrm{e}_{x,i:i+}^{-1}(\gamma_i)$. If $g_1, g_2 \in [\![\gamma; x, r]\!]_{\mathbf{G}(E)}^{(s)}$ and $g_j = \prod_{0<i<r} g_{ji}$ are decompositions as in (5.2.2) for $j = 1, 2$, then put

$$
(5.2.4) \qquad \log_\Lambda \sqrt{\mathcal{B}}(g_1, g_2) = \tfrac{1}{2} \sum_{0<i<r} \overline{X^*\big[[Y_i, X_{2i}], X_{1i}\big]},
$$

where $t \mapsto \overline{t}$ is the natural map from E_0 to $\mathfrak{f}_E$, and $X_{ji} \in \big(\mathrm{e}_{x,((r-i)/2):((r-i)/2)+}^E\big)^{-1}(g_{ji}) \cap \mathrm{Lie}(C_{\mathbf{G}}^{(i)}(\gamma))(E)_{x,(r-i)/2}$ for $0 < i < r$ and $j = 1, 2$. Note that there do exist elements X_{ji} in the indicated intersection, by Lemma A.4; and that this definition does not depend on the choices of the various Y_i and X_{ji}.

Lemma 5.2.8. $\log_\Lambda \sqrt{\mathcal{B}}$ *and* $\mathcal{B}$ *are symmetric, and*

$$
\mathcal{B}(g_1 g'_1, g_2) = \mathcal{B}(g_1, g_2)\mathcal{B}(g'_1, g_2),
$$
$$
\log_\Lambda \sqrt{\mathcal{B}}(g_1 g'_1, g_2) = \log_\Lambda \sqrt{\mathcal{B}}(g_1, g_2) + \log_\Lambda \sqrt{\mathcal{B}}(g'_1, g_2),
$$

and

$$
\big(\Lambda(\log_\Lambda \sqrt{\mathcal{B}}(g_1, g_2))\big)^2 = \mathcal{B}(g_1, g_2)
$$

for $g_1, g'_1, g_2 \in [\![\gamma; x, r]\!]_{\mathbf{G}(E)}^{(s)}$.

Of course, $\mathcal{B}$ and $\log_\Lambda \sqrt{\mathcal{B}}$ exhibit analogous behaviors in the second variable, by symmetry.

Proof. The symmetry and multiplicativity of $\mathcal{B}$ will follow from those of $\log_\Lambda \sqrt{\mathcal{B}}$.

By Lemma 5.2.7 and (5.2.4), it suffices to show the desired facts on each $C_{\mathbf{G}}^{(i)}(\gamma)(E)_{x,(r-i)/2}$. Accordingly, fix $0 < i < r$ and $g_1, g_2 \in C_{\mathbf{G}}^{(i)}(\gamma)(E)_{x,(r-i)/2}$. Let $X_j \in \mathrm{Lie}(C_{\mathbf{G}}^{(i)}(\gamma))(E)_{x,(r-i)/2}$ satisfy $g_j \in \mathsf{e}_{x,((r-i)/2):((r-i)/2)+}^E(X_j)$ for $j = 1, 2$. Then

$$\big[[Y_i, X_1], X_2\big] = \big[[Y_i, X_2], X_1\big] + \big[Y_i, [X_1, X_2]\big]$$

by the Jacobi identity.

Since $\mathbf{T}$ is a maximal torus in $C_{\mathbf{G}}^{(i)}(\gamma)$, we have $\gamma_i \in Z_G^{(i)}(\gamma) \subseteq T$. Since $\big[\gamma_i, [g_1, g_2]\big] \in C_{\mathbf{G}}^{(i)}(\gamma)(E)_{x,r}$, Lemma 5.30 of [6] gives $\big[\gamma_i, [g_1, g_2]\big] \in (\mathbf{T}, C_{\mathbf{G}}^{(i)}(\gamma))(E)_{x,(r+,r)}$. By two applications of Hypothesis A.1(5), we have that $\big[Y_i, [X_1, X_2]\big] \in (\mathsf{e}_{x,r:r+}^E)^{-1}\big[\gamma_i, [g_1, g_2]\big]$, hence (by Hypothesis A.1(6)) that

$$\big[Y_i, [X_1, X_2]\big] \in \mathrm{Lie}(\mathbf{T}, C_{\mathbf{G}}^{(i)}(\gamma))(E)_{x,(r+,r)}.$$

By an easy analogue of Proposition 5.40 of [6], $\big[Y_i, [X_1, X_2]\big] \in \mathrm{Lie}(\mathbf{T})(E)_{r+} \oplus (\mathrm{Lie}(\mathbf{T})^{\perp}(E) \cap \mathrm{Lie}(\mathbf{G})(E)_{x,r})$, where $\mathrm{Lie}(\mathbf{T})^{\perp} = \bigoplus_{\alpha \in \Phi(\mathbf{G},\mathbf{T})} \mathrm{Lie}(\mathbf{G})_\alpha$. Since $X^* \in \mathfrak{z}(\mathfrak{g}')^* + \mathfrak{g}_{x,r+}^* \subseteq \mathfrak{t}^* + \mathfrak{g}_{x,r+}^*$, we have that $X^*\big[Y_i, [X_1, X_2]\big] \in E_{0+}$, so

$$\log_{\Lambda} \sqrt{\mathcal{B}}(g_2, g_1) = \tfrac{1}{2}\overline{X^*\big[[Y_i, X_1], X_2\big]} = \tfrac{1}{2}\overline{X^*\big[[Y_i, X_2], X_1\big]} = \log_{\Lambda} \sqrt{\mathcal{B}}(g_1, g_2).$$

Now fix $g_1' \in C_{\mathbf{G}}^{(i)}(\gamma)(E)_{x,(r-i)/2}$, and choose $X_1' \in \mathrm{Lie}(C_{\mathbf{G}}^{(i)}(\gamma))(E)_{x,(r-i)/2}$ such that $g_1' \in \mathsf{e}_{x,((r-i)/2):((r-i)/2)+}^E(X_1')$. Then, since $\mathsf{e}_{x,((r-i)/2):((r-i)/2)+}^E$ is a homomorphism, we have that $g_1 g_1' \in \mathsf{e}_{x,((r-i)/2):((r-i)/2)+}^E(X_1 + X_1')$, and it follows immediately that $\log_{\Lambda} \sqrt{\mathcal{B}}(g_1 g_1', g_2) = \log_{\Lambda} \sqrt{\mathcal{B}}(g_1, g_2) \log_{\Lambda} \sqrt{\mathcal{B}}(g_1', g_2)$.

Finally, note that $\big[[g_2, \gamma^{-1}], g_1\big] = \big[[g_2, \gamma_{\geq i}^{-1}], g_1\big] \equiv \big[[g_2, \gamma_i^{-1}], g_1\big] \pmod{G_{x,r+}}$ and, by two applications of Hypothesis A.1(5) and the fact that $\gamma_i^{-1} \in \mathsf{e}_{x,r:r+}^E(Y_i)$, that $\big[[g_2, \gamma_i^{-1}], g_1\big] \in \mathsf{e}_{x,r:r+}^E\big(\big[[Y_i, X_2], X_1\big]\big)$, so

$$\mathcal{B}(g_1, g_2) = \hat{\phi}_E\big(\big[[g_2, \gamma^{-1}], g_1\big]\big) = \Lambda\big(X^*\big[[Y_i, X_2], X_1\big]\big)$$

$$= \Big(\Lambda\big(\tfrac{1}{2}X^*\big[[Y_i, X_2], X_1\big]\big)\Big)^2 = \big(\Lambda(\log_{\Lambda} \sqrt{\mathcal{B}}(g_1, g_2))\big)^2. \quad \square$$

Corollary 5.2.9. *We have $\mathcal{Q}(g) = \Lambda(\log_{\Lambda} \sqrt{\mathcal{B}}(g, g))$ for $g \in [\![\gamma; x, r]\!]_{\mathbf{G}(E)}^{(s)}$.*

Proof. Note that $\gamma_i \in Z_G^{(r)}(\gamma) \subseteq T$ for $0 < i < r$. Since the choice of $Y_i \in \mathsf{e}_{x,i:i+}^{-1}(\gamma_i)$ does not matter in (5.2.4), by Lemma A.4, we may, and do, take $Y_i \in \mathfrak{t}$. If $0 < i < r$, $\alpha \in \widetilde{\Phi}(C_{\mathbf{G}}^{(i)}(\gamma), \mathbf{T})$, and $g \in \mathbf{U}_\alpha(E) \cap \mathbf{G}(E)_{x,(r-i)/2}$, then choose

$$X \in \mathrm{Lie}(\mathbf{U}_\alpha)(E) \cap \mathrm{Lie}(\mathbf{G})(E)_{x,(r-i)/2}$$

such that

$$g \in \mathsf{e}_{x,((r-i)/2):((r-i)/2)+}^E(X).$$

(Such an X exists, by Lemma A.6.) Since $\mathrm{Lie}(\mathbf{U}_\alpha)$ is Abelian and preserved by $\mathrm{ad}(Y_i)$, we have that $\big[[Y_i, X], X\big] = 0$, hence that

$$\log_\Lambda \sqrt{\mathcal{B}}(g, g) = \tfrac{1}{2}\overline{X^*\big[[Y_i, X], X\big]} = 0.$$

If $\alpha = 0$, then $[\gamma_{<r}^{-1}, g] = 1 \in \ker \hat{\phi}_E$. Otherwise, by Lemma 4.1, Remark 4.2, and Hypothesis 2.3, we have

$$[\gamma_{<r}^{-1}, g] \in \mathbf{U}_\alpha(E) \cap \mathbf{G}(E)_{x,(r+i)/2} \subseteq (\mathbf{T}, \mathbf{G})(E)_{x,(r+,s+)} \subseteq \ker \hat{\phi}_E.$$

In either case,

$$\mathcal{Q}(g) = \hat{\phi}_E([\gamma^{-1}, g]) = \hat{\phi}_E([\gamma_{<r}^{-1}, g]) = 1 = \Lambda(\log_\Lambda \sqrt{\mathcal{B}}(g, g)).$$

By Definitions 5.13 and 9.3 of [6], we have shown equality on a set of semigroup generators for $[\![\gamma; x, r]\!]_{\mathbf{G}(E)}^{(s)}$. Since $g \mapsto \mathcal{Q}(g)\Lambda(\log_\Lambda \sqrt{\mathcal{B}}(g, g))^{-1}$ is multiplicative by (5.2.1) and Lemma 5.2.8, we have equality everywhere. $\qquad\square$

Lemma 5.2.10. *If $g_2 \in [\![\gamma; x, r]\!]^{(s)}$ is such that $\mathcal{B}(g, g_2) = 1$ for all $g \in [\![\gamma; x, r]\!]^{(s)}$, then $g_2 \in [\![\gamma; x, r]\!]_{G'}^{(s)}[\![\gamma_{<r}; x, r+]\!]^{(s)}$.*

As remarked after Corollary 5.2.6, the unexpected appearance of $[\![\gamma_{<r}; x, r+]\!]^{(s)}$ in place of $[\![\gamma; x, r+]\!]^{(s)}$ compensates for the fact that γ might not have a normal $(r+)$-approximation.

Proof. By Lemma 5.2.7 (and the fact that, by Proposition 5.40 of [6], all the terms in a decomposition (5.2.2) of an F-rational element may be taken to be F-rational), it suffices to consider the case that $g_2 \in C_G^{(i)}(\gamma)_{x,(r-i)/2}$ for some $0 < i < r$. Notice that $[g_2, \gamma^{-1}] = [g_2, \gamma_{\geq i}^{-1}] \in C_G^{(i)}(\gamma)_{x,(r+i)/2}$. Therefore, if $g \in (C_G^{(i)}(\gamma), G)_{x,(((r-i)/2)+,(r-i)/2)}$, then Lemma 5.32 of [6] gives that $\big[[g_2, \gamma^{-1}], g\big] \in (C_G^{(i)}(\gamma), G)_{x,(r+,r)}$, hence, by Lemma 4.1 and Remark 4.2, that $\big[[\gamma^{-1}, g_2], \hat{\phi}\big](g) := \hat{\phi}\big([[g_2, \gamma^{-1}], g]\big) = 1$. That is, $\big[[\gamma^{-1}, g_2], \hat{\phi}\big]$ is trivial on $(C_G^{(i)}(\gamma), G)_{x,(r+,r)}$. By hypothesis, it is also trivial on $C_G^{(i)}(\gamma)_{x,r}$, hence, by Proposition 5.40 of [6], on $G_{x,r}$. By Lemma B.1,

$$g_2 \in (C_{G'}^{(i)}(\gamma), C_G^{(i)}(\gamma))_{x,((r-i)/2,((r-i)/2)+)} \subseteq [\![\gamma; x, r]\!]_{G'}^{(s)}[\![\gamma_{<r}; x, r+]\!]^{(s)},$$

as desired. $\qquad\square$

Via (5.2.1), we may view Lemma 5.2.10 as an "upper bound" on the size of the level set of $\mathcal{Q}$ containing a fixed element g. Since Corollary 5.2.6 describes uniform "lower bounds" on the sizes of the level sets of $\mathcal{Q}$, we have quite precise local constancy information.

Notation 5.2.11. In addition to Notation 3.6, for $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, let $\mathbf{V}_\alpha$ denote the image of

$$\mathrm{Lie}(\mathbf{G})_\alpha(E) \cap \mathrm{Lie}(\llbracket \gamma; x, r \rrbracket^{(s)}_{\mathbf{G}(E)})$$

in

$$\mathrm{Lie}(\llbracket \gamma; x, r \rrbracket^{(s)}_{\mathbf{G}'(E)} \llbracket \gamma_{<r}; x, r+ \rrbracket^{(s)}_{\mathbf{G}(E)}) \backslash \mathrm{Lie}(\llbracket \gamma; x, r \rrbracket^{(s)}_{\mathbf{G}(E)}),$$

and V_α the set of Γ_α-fixed points in $\mathbf{V}_\alpha$. (The symbols $\mathbf{V}_\alpha$ and V_α had a different meaning in §3.) More concretely, we have that $\mathbf{V}_\alpha = \{0\}$ if $\alpha \in \Phi(\mathbf{G}', \mathbf{T}) \cup \Phi(C_{\mathbf{G}}^{(r)}(\gamma), \mathbf{T})$ or $\alpha \notin \Phi(C_{\mathbf{G}}^{(0+)}(\gamma), \mathbf{T})$; and, if $\alpha \in \Phi(C_{\mathbf{G}}^{(0+)}(\gamma), \mathbf{T}) \backslash (\Phi(\mathbf{G}', \mathbf{T}) \cup \Phi(C_{\mathbf{G}}^{(r)}(\gamma), \mathbf{T}))$, then $\mathbf{V}_\alpha \cong {}_E\mathfrak{u}_{(\alpha+(r-i)/2):(\alpha+(r-i)/2)+}$, where $i = \mathrm{ord}(\alpha(\gamma_{<r}) - 1)$ and $\alpha + (r-i)/2$ is the affine root with gradient α whose value at x is $(r-i)/2$. Put $\Upsilon(\phi, \gamma) = \{ \alpha \in \Phi(\mathbf{G}, \mathbf{T}) \mid V_\alpha \neq \{0\} \}$, and

$$\Upsilon_{\mathrm{symm,unram}}(\phi, \gamma) = \{ \alpha \in \Upsilon(\phi, \gamma) \mid -\alpha \in \Gamma \cdot \alpha \text{ and } \eta_\alpha \neq 1 \text{ on } \mathfrak{f}_\alpha \},$$

$$\Upsilon_{\mathrm{symm,ram}}(\phi, \gamma) = \{ \alpha \in \Upsilon(\phi, \gamma) \mid -\alpha \in \Gamma \cdot \alpha \text{ and } \eta_\alpha = 1 \text{ on } \mathfrak{f}_\alpha \},$$

and

$$\Upsilon^{\mathrm{symm}}(\phi, \gamma) = \{ \alpha \in \Upsilon(\phi, \gamma) \mid -\alpha \notin \Gamma \cdot \alpha \}.$$

We will omit ϕ and γ from the notation when convenient. Note that all of these sets are $\Gamma \times \{\pm 1\}$-stable. We denote by $\dot\Upsilon_{\mathrm{symm,unram}}(\phi, \gamma)$ and $\dot\Upsilon_{\mathrm{symm,ram}}(\phi, \gamma)$ sets of representatives for the Γ-orbits in the appropriate sets; and by $\ddot\Upsilon^{\mathrm{symm}}(\phi, \gamma)$ a set of representatives for the $\Gamma \times \{\pm 1\}$-orbits in Υ^{symm}. Finally, put $\dot\Upsilon_{\mathrm{symm}}(\phi, \gamma) = \dot\Upsilon_{\mathrm{symm,unram}} \cup \dot\Upsilon_{\mathrm{symm,ram}}$, $\dot\Upsilon(\phi, \gamma) = \dot\Upsilon_{\mathrm{symm}} \cup \pm\ddot\Upsilon^{\mathrm{symm}}$, and $f(\dot\Upsilon_{\mathrm{symm,ram}}(\phi, \gamma)) = \sum_{\alpha \in \dot\Upsilon_{\mathrm{symm,ram}}} f_\alpha$.

The proof of the following result is a relatively straightforward application of results from [6] that allow us to combine and manipulate groups of the form ${}_\mathbf{T}G_{x,f}$.

Proposition 5.2.12.

$$|\widetilde{\mathfrak{G}}(\phi, \gamma)| = \left[\llbracket \gamma_{<r}; x, r \rrbracket : \llbracket \gamma_{<r}; x, r \rrbracket_{G'} G_{x,s} \right]^{1/2}$$

$$\times \left[\llbracket \gamma_{<r}; x, r+ \rrbracket : \llbracket \gamma_{<r}; x, r+ \rrbracket_{G'} G_{x,s+} \right]^{1/2}$$

$$\times |(C_{G'}^{(0+)}(\gamma), C_G^{(0+)}(\gamma))_{x,(r,s):(r,s+)}|^{-1/2}$$

Proof. Since none of the quantities involved change if we replace γ by $\gamma_{<r}$, we do so. Write $\mathcal{V} = \llbracket \gamma; x, r \rrbracket^{(s)}_{G'} \llbracket \gamma; x, r+ \rrbracket^{(s)} \backslash \llbracket \gamma; x, r \rrbracket^{(s)}$. For $0 < j < s$, we have that $(C_{G'}^{(r-2j)}(\gamma), C_G^{(r-2j)}(\gamma))_{x,(j,j):(j,j+)}$ is naturally isomorphic to a subgroup of $\mathcal{V}$. It is straightforward to check that $\mathcal{V}$ is the direct sum of

these subgroups. Then reasoning as in the proof of Proposition 3.8 (applied to each direct summand) shows that

$$(5.2.5) \qquad \mathcal{V} \cong \bigoplus_{\alpha \in \dot{\Upsilon}_{\mathrm{symm}}} V_\alpha \oplus \bigoplus_{\alpha \in \ddot{\Upsilon}^{\mathrm{symm}}} V_{\pm\alpha} =: V,$$

where, as before, we have written $V_{\pm\alpha} = V_\alpha \oplus V_{-\alpha}$ for $\alpha \in \ddot{\Upsilon}^{\mathrm{symm}}$. Write B for the pairing on V induced by (5.2.5) (and the pairing $\log_\Lambda \sqrt{\mathcal{B}}$ on $\mathcal{V}$). Notice that B is $\mathfrak{f}$-bilinear. By Lemma 5.2.10, B is non-degenerate.

By Corollaries 5.2.6 and 5.2.9,

$$\widetilde{\mathfrak{G}} = \left[[\![\gamma; x, r]\!]^{(s)}_{G'} [\![\gamma; x, r+]\!] : [\![\gamma; x, r]\!]^{(s)}_{G'} C_G^{(0+)}(\gamma)_{x,s} \right] \sum_{g \in \mathcal{V}} \mathcal{Q}(g)$$

$$= \left[[\![\gamma; x, r]\!]^{(s)}_{G'} [\![\gamma; x, r+]\!] : [\![\gamma; x, r]\!]^{(s)}_{G'} C_G^{(0+)}(\gamma)_{x,s} \right] \sum_{X \in V} \Lambda(B(X, X)).$$

Since B is non-degenerate, we have by Lemma 5.2.2 that (in the notation of that lemma)

$$\left| \sum_{X \in V} \Lambda(B(X, X)) \right| = \left| |V|^{1/2} \, \mathfrak{G}(V, B) \right| = |V|^{1/2} = |\mathcal{V}|^{1/2}.$$

Thus

$$(5.2.6) \quad
\begin{aligned}
|\widetilde{\mathfrak{G}}| &= \left[[\![\gamma; x, r]\!]^{(s)}_{G'} [\![\gamma; x, r+]\!]^{(s)} : [\![\gamma; x, r]\!]^{(s)}_{G'} C_G^{(0+)}(\gamma)_{x,s} \right] \\
&\qquad \times \left[[\![\gamma; x, r]\!]^{(s)} : [\![\gamma; x, r]\!]^{(s)}_{G'} [\![\gamma; x, r+]\!]^{(s)} \right]^{1/2} \\
&= \left[[\![\gamma; x, r]\!]^{(s)}_{G'} [\![\gamma; x, r+]\!]^{(s)} : [\![\gamma; x, r]\!]^{(s)}_{G'} C_G^{(0+)}(\gamma)_{x,s} \right]^{1/2} \\
&\qquad \times \left[[\![\gamma; x, r]\!]^{(s)} : [\![\gamma; x, r]\!]^{(s)}_{G'} C_G^{(0+)}(\gamma)_{x,s} \right]^{1/2}.
\end{aligned}$$

(In the second and third expressions being compared, the first terms are the same, except that the latter contains an extra exponent of $1/2$.) By Remarks 6.7(1) and 6.7(4) of [6],

$$(5.2.7) \qquad [\![\gamma; x, r]\!]^{(s)} G_{x,s} = [\![\gamma; x, r]\!].$$

Upon writing $[\![\gamma; x, r+]\!]^{(s)}$ and $[\![\gamma; x, r+]\!]^{(s+)}$ as groups of the form $_{\mathbf{T}}G_{x,f}$, using Definition 9.3 of *loc. cit.*, we see that $[\![\gamma; x, r+]\!]^{(s)} = [\![\gamma; x, r+]\!]^{(s+)}$. Therefore, again by Remarks 6.7(1) and 6.7(4) of *loc. cit.*, we have that

$$(5.2.8) \qquad [\![\gamma; x, r+]\!]^{(s)} G_{x,s+} = [\![\gamma; x, r+]\!].$$

By Proposition 5.40 and Lemma 5.29 of [6], we have the following equalities:

(5.2.9)
$$\llbracket \gamma; x, r \rrbracket^{(s)} \cap \llbracket \gamma; x, r \rrbracket_{G'} G_{x,s} = \llbracket \gamma; x, r \rrbracket_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \,,$$

(5.2.10)
$$\llbracket \gamma; x, r+ \rrbracket^{(s)} \cap \llbracket \gamma; x, r \rrbracket_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} = \llbracket \gamma; x, r+ \rrbracket_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \,,$$

(5.2.11)
$$\llbracket \gamma; x, r+ \rrbracket^{(s)} \cap \llbracket \gamma; x, r+ \rrbracket_{G'} (C_G^{(0+)}(\gamma), G)_{x,(s,s+)} = \llbracket \gamma; x, r+ \rrbracket_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \,,$$

and

(5.2.12)
$$(C_{G'}^{(0+)}(\gamma), C_G^{(0+)}(\gamma))_{x,(r,s)} \cap \llbracket \gamma; x, r \rrbracket_{G'} G_{x,s+} = (C_{G'}^{(0+)}(\gamma), C_G^{(0+)}(\gamma))_{x,(r,s+)} .$$

By (5.2.7) and (5.2.9), we have a bijection

$$\llbracket \gamma; x, r \rrbracket_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \backslash \llbracket \gamma; x, r \rrbracket^{(s)} \longrightarrow \llbracket \gamma; x, r \rrbracket_{G'} G_{x,s} \backslash \llbracket \gamma; x, r \rrbracket,$$

so

(5.2.13)
$$\left[\llbracket \gamma; x, r \rrbracket^{(s)} : \llbracket \gamma; x, r \rrbracket_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \right] = \left[\llbracket \gamma; x, r \rrbracket : \llbracket \gamma; x, r \rrbracket_{G'} G_{x,s} \right].$$

By (5.2.8), since $G_{x,s+} \subseteq \llbracket \gamma; x, r+ \rrbracket_{G'} (C_G^{(0+)}(\gamma), G)_{x,(s,s+)}$, we have that

$$\llbracket \gamma; x, r+ \rrbracket_{G'} (C_G^{(0+)}(\gamma), G)_{x,(s,s+)} \backslash \llbracket \gamma; x, r+ \rrbracket$$

is naturally in bijection with

$$\left(\llbracket \gamma; x, r+ \rrbracket_{G'} (C_G^{(0+)}(\gamma), G)_{x,(s,s+)} \cap \llbracket \gamma; x, r+ \rrbracket^{(s)} \right) \backslash \llbracket \gamma; x, r+ \rrbracket^{(s)},$$

which, by (5.2.11), is just

$$\llbracket \gamma; x, r+ \rrbracket_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \backslash \llbracket \gamma; x, r+ \rrbracket^{(s)}.$$

By (5.2.10), this latter set is naturally in bijection with

$$\llbracket \gamma; x, r \rrbracket_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \backslash \llbracket \gamma; x, r \rrbracket_{G'}^{(s)} \llbracket \gamma; x, r+ \rrbracket^{(s)}.$$

Thus,

(5.2.14) $\left[\llbracket \gamma; x, r \rrbracket_{G'}^{(s)} \llbracket \gamma; x, r+ \rrbracket^{(s)} : \llbracket \gamma; x, r \rrbracket_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \right]$

$$= \left[\llbracket \gamma; x, r+ \rrbracket : \llbracket \gamma; x, r+ \rrbracket_{G'} (C_G^{(0+)}(\gamma), G)_{x,(s,s+)} \right].$$

By (5.2.12), we have an injection

$$(C_{G'}^{(0+)}(\gamma), C_G^{(0+)}(\gamma))_{x,(r,s):(r,s+)} \longrightarrow \llbracket \gamma; x, r+ \rrbracket_{G'} G_{x,s+} \backslash \llbracket \gamma; x, r+ \rrbracket.$$

By Proposition 5.40 of [6], the cokernel of this injection is

$$\llbracket \gamma; x, r+ \rrbracket_{G'} (C_G^{(0+)}(\gamma), G)_{x,(s,s+)} \backslash \llbracket \gamma; x, r+ \rrbracket,$$

so

$$(5.2.15) \quad \left[\llbracket \gamma; x, r+ \rrbracket : \llbracket \gamma; x, r+ \rrbracket_{G'} G_{x,s+} \right]$$
$$= |(C_{G'}^{(0+)}(\gamma), C_G^{(0+)}(\gamma))_{x,(r,s):(r,s+)}|$$
$$\times \left[\llbracket \gamma; x, r+ \rrbracket : \llbracket \gamma; x, r+ \rrbracket_{G'} (C_G^{(0+)}(\gamma), G)_{x,(s,s+)} \right]$$
$$= |(C_{G'}^{(0+)}(\gamma), C_G^{(0+)}(\gamma))_{x,(r,s):(r,s+)}|$$
$$\times \left[\llbracket \gamma; x, r \rrbracket_{G'}^{(s)} \llbracket \gamma; x, r+ \rrbracket^{(s)} : \llbracket \gamma; x, r \rrbracket_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \right].$$

(where the last equality follows from (5.2.14)). Upon plugging (5.2.13) and (5.2.15) into (5.2.6), we obtain the desired formula for $|\tilde{\mathfrak{G}}|$. $\qquad\square$

Proposition 5.2.13.

$$\mathfrak{G}(\phi, \gamma) = (-1)^{|\dot{\Upsilon}_{\mathrm{symm}}(\phi,\gamma)|} (-\mathfrak{G}_\Lambda(\mathfrak{f}))^{f(\dot{\Upsilon}_{\mathrm{symm,ram}}(\phi,\gamma))}$$
$$\times \prod_{\alpha \in \dot{\Upsilon}_{\mathrm{symm,ram}}(\phi,\gamma)} \left[\mathrm{sgn}_{\mathfrak{f}_\alpha} \left(\tfrac{1}{2} e_\alpha N_{F_\alpha/F_{\pm\alpha}}(w_\alpha) d\alpha^\vee(X^*)(\alpha(\gamma_{<r}) - 1) \right) \right.$$
$$\left. \times \mathrm{sgn}_{F_{\pm\alpha}}(\mathbf{G}_{\pm\alpha}) \right],$$

where

- $\mathfrak{G}_\Lambda(\mathfrak{f})$ *is as in Definition 5.2.1;*
- $d\alpha^\vee(X^*) = X^*(d\alpha^\vee(1))$*;*
- w_α *is any element of F_α, of valuation $\left(r - \mathrm{ord}(\alpha(\gamma_{<r}) - 1) \right)/2$, whose square lies in $F_{\pm\alpha}$;*
- $\mathbf{G}_{\pm\alpha}$ *is the group generated by the root subgroups $\mathbf{U}_\alpha$ and $\mathbf{U}_{-\alpha}$ of $\mathbf{G}$; and*
- $\mathrm{sgn}_{F_{\pm\alpha}}(\mathbf{G}_{\pm\alpha})$ *is $+1$ or -1 according as $\mathbf{G}_{\pm\alpha}$ is or is not $F_{\pm\alpha}$-split, respectively.*

We will show in the proof that an element w_α as in the statement exists.

Proof. As in the proof of Proposition 5.2.12, we may, and do, replace γ by $\gamma_{<r}$. Recall the notation B and V from the proof of Proposition 5.2.12, and the elements Y_i chosen before Lemma 5.2.8. By the way we defined $\log_\Lambda \sqrt{\mathcal{B}}$ and the isomorphism in (5.2.5), V_α is B-orthogonal to V_β unless $-\beta \in \Gamma \cdot \alpha$, and

$$(5.2.16) \quad B(X, X') = \sum_{\eta \in \Gamma/\Gamma_\alpha} \eta \left(\tfrac{1}{2} X^* \left[[Y_i, X], \eta_\alpha X' \right] \right)$$

$$\text{for } X, X' \in V_\alpha \text{ with } \alpha \in \dot{\Upsilon}_{\mathrm{symm}},$$

where $i = \mathrm{ord}(\alpha(\gamma) - 1)$. In particular, the sums on the right-hand side of (5.2.5) are B-orthogonal.

Put $\mathfrak{G} = \mathfrak{G}(\phi, \gamma)$. We showed in the proof of Proposition 5.2.12 that $\mathfrak{G} = \mathfrak{G}(V, B)$, in the notation of Lemma 5.2.2; so, by that lemma, we have that $\mathfrak{G} = \mathrm{sgn}_{\mathfrak{f}}(\det B)\mathfrak{G}_{\Lambda}(\mathfrak{f})^{\dim_{\mathfrak{f}} V}$. By (5.2.5),

(†)
$$\mathfrak{G} = \prod_{\alpha \in \dot{\Upsilon}_{\mathrm{symm}}} \mathrm{sgn}_{\mathfrak{f}}(\det B\big|_{V_\alpha})\mathfrak{G}_{\Lambda}(\mathfrak{f})^{\dim_{\mathfrak{f}} V_\alpha} \cdot \prod_{\alpha \in \ddot{\Upsilon}_{\mathrm{symm}}} \mathrm{sgn}_{\mathfrak{f}}(\det B\big|_{V_{\pm\alpha}})\mathfrak{G}_{\Lambda}(\mathfrak{f})^{\dim_{\mathfrak{f}} V_{\pm\alpha}}.$$

We will use in our calculations below the fact that $\mathfrak{G}_{\Lambda}(\mathfrak{f})^2 = \mathrm{sgn}_{\mathfrak{f}}(-1)$.

For $\alpha \in \ddot{\Upsilon}^{\mathrm{symm}}$, the matrix of the restriction of B to $V_{\pm\alpha}$, with respect to a suitable basis, is of the form $\begin{pmatrix} 0 & M \\ M^{\mathrm{t}} & 0 \end{pmatrix}$ for some matrix M. Thus the determinant of this restriction is in the square class of $(-1)^{\dim_{\mathfrak{f}} V_{\pm\alpha}/2}$, so

(5.2.17) $$\mathrm{sgn}_{\mathfrak{f}}(\det B\big|_{V_{\pm\alpha}})\mathfrak{G}_{\Lambda}(\mathfrak{f})^{\dim_{\mathfrak{f}} V_{\pm\alpha}} = 1.$$

Fix $\alpha \in \dot{\Upsilon}_{\mathrm{symm}}$ and put $i = \mathrm{ord}(\alpha(\gamma)-1)$. Since $\alpha \in \Phi(C_{\mathbf{G}}^{(0+)}(\gamma), \mathbf{T})$, we have that $i > 0$. As in the proof of Corollary 5.2.9, we may, and do, assume that $Y_i \in \mathfrak{t}$. As in the proof of Proposition 3.8, we have an isomorphism $\iota_\alpha : V_\alpha \cong \mathfrak{f}_\alpha$. Put $X_\alpha = \iota_\alpha^{-1}(1)$ and $c_\alpha = \overline{X^*\big[[Y_i, X_\alpha], \eta_\alpha X_\alpha\big]} \in \mathfrak{f}_{\pm\alpha}$. By (5.2.16), ι_α identifies the restriction of B to V_α with the pairing

$$(t_1, t_2) \mapsto \tfrac{1}{2}e_\alpha \, \mathrm{tr}_{\mathfrak{f}_\alpha/\mathfrak{f}}(c_\alpha \cdot t_1\eta_\alpha(t_2))$$

on $\mathfrak{f}_\alpha$. The determinant of this pairing is

(∗) $$\left(\tfrac{1}{2}e_\alpha\right)^{f_\alpha} N_{\mathfrak{f}_\alpha/\mathfrak{f}}(c_\alpha)\Delta,$$

where Δ is the determinant of $(t_1, t_2) \mapsto \mathrm{tr}_{\mathfrak{f}_\alpha/\mathfrak{f}}(t_1\eta_\alpha(t_2))$.

If $\alpha \in \dot{\Upsilon}_{\mathrm{symm,unram}}$, then $f_\alpha = \dim_{\mathfrak{f}} V_\alpha$ is even (so $\left(\tfrac{1}{2}e_\alpha\right)^{f_\alpha}$ is a square) and η_α is a $(\dim_{\mathfrak{f}} V_\alpha/2)$th power of a generator of $\mathrm{Gal}(\mathfrak{f}_\alpha/\mathfrak{f})$, so $\mathrm{sgn}_{\mathrm{Gal}(\mathfrak{f}_\alpha/\mathfrak{f})}(\eta_\alpha) = (-1)^{\dim_{\mathfrak{f}} V_\alpha/2}$ (where $\mathrm{sgn}_{\mathrm{Gal}(\mathfrak{f}_\alpha/\mathfrak{f})}$ is as in Lemma 3.1). Further, we have that $N_{\mathfrak{f}_\alpha/\mathfrak{f}}(c_\alpha) = N_{\mathfrak{f}_{\pm\alpha}/\mathfrak{f}}(c_\alpha)^2 \in (\mathfrak{f}^\times)^2$. By Lemma 3.1,

$$\mathrm{sgn}_{\mathfrak{f}}(\Delta) = \left(-\mathrm{sgn}_{\mathfrak{f}}(-1)^{\dim_{\mathfrak{f}} V_\alpha/2}\right)^{f_\alpha+1} = -\mathrm{sgn}_{\mathfrak{f}}(-1)^{\dim_{\mathfrak{f}} V_\alpha/2},$$

so (∗) gives

(5.2.18) $$\mathrm{sgn}_{\mathfrak{f}}(\det B\big|_{V_\alpha})\mathfrak{G}_{\Lambda}(\mathfrak{f})^{\dim_{\mathfrak{f}} V_\alpha} = \mathrm{sgn}_{\mathfrak{f}}(\Delta)(\mathfrak{G}_{\Lambda}(\mathfrak{f})^2)^{\dim_{\mathfrak{f}} V_\alpha/2}$$
$$= -\mathrm{sgn}_{\mathfrak{f}}(-1)^{\dim_{\mathfrak{f}} V_\alpha/2} \, \mathrm{sgn}_{\mathfrak{f}}(-1)^{\dim_{\mathfrak{f}} V_\alpha/2} = -1.$$

If $\alpha \in \dot{\Upsilon}_{\mathrm{symm,ram}}$, then $F_\alpha/F_{\pm\alpha}$ is totally ramified. Since $\eta_\alpha \alpha(\gamma) = \alpha(\gamma)^{-1}$, we have that $i = \mathrm{ord}(\alpha(\gamma)-1) \in \mathrm{ord}(F_\alpha^\times) \smallsetminus \mathrm{ord}(F_{\pm\alpha}^\times)$. Similarly, since $\eta_\alpha d\alpha^\vee(X^*) = -d\alpha^\vee(X^*)$, we have that $-r \in \mathrm{ord}(F_\alpha^\times) \smallsetminus \mathrm{ord}(F_{\pm\alpha}^\times)$. Since $\mathrm{ord}(F_\alpha^\times)/\mathrm{ord}(F_{\pm\alpha}^\times) \cong \mathbb{Z}/2\mathbb{Z}$, we have that $r - i \in \mathrm{ord}(F_{\pm\alpha}^\times)$, so $(r - i)/2 \in \mathrm{ord}(F_\alpha^\times)$. Let ϖ_α be a uniformizer of F_α that is negated by η_α, and let w_α be a power of ϖ_α that has valuation $(r - i)/2$. In particular, $w_\alpha^2 \in F_{\pm\alpha}$.

Put $H_\alpha = d\alpha^\vee(1) \in \mathrm{Lie}(\mathbf{G}_{\pm\alpha})(F_\alpha)$, so that $d\alpha^\vee(X^*) = X^*(H_\alpha)$. Then $\mathrm{Lie}(\mathbf{G}_{\pm\alpha})$ is the sum of the α-weight space $\mathrm{Lie}(\mathbf{G})_\alpha$, the $(-\alpha)$-weight space $\mathrm{Lie}(\mathbf{G})_{-\alpha}$, and the Cartan subgroup $\mathbf{t}^\alpha$ spanned by H_α. Since $[X_\alpha, \eta_\alpha X_\alpha]$ and H_α both belong to the 1-dimensional F_α-space $\mathbf{t}^\alpha(F_\alpha)$ and $H_\alpha \neq 0$, we have that there is a constant $t_\alpha \in F_\alpha$ so that $[X_\alpha, \eta_\alpha X_\alpha] = t_\alpha H_\alpha$. Since H_α and $[X_\alpha, \eta_\alpha X_\alpha]$ are both negated by η_α, we have that $t_\alpha \in F_{\pm\alpha}$. Then

$$c_\alpha = \overline{X^*\big[[Y_i, X_\alpha], \eta_\alpha X_\alpha\big]} = \overline{X^*(d\alpha(Y_i)t_\alpha H_\alpha)} = \overline{t_\alpha d\alpha^\vee(X^*)d\alpha(Y_i)}.$$

We claim that $\mathrm{ord}(t_\alpha) = r - i$. Indeed, since $\mathrm{d}_x(H_\alpha) = 0$, we have that $\mathrm{ord}(t_\alpha) = \mathrm{d}_x([X_\alpha, \eta_\alpha X_\alpha]) \geq r - i$. Suppose that we had $\mathrm{ord}(t_\alpha) > r - i$. Since $\mathrm{ord}(d\alpha^\vee(X^*)) \geq -r$ and $\mathrm{ord}(d\alpha(Y_i)) \geq i$, this would mean that c_α was the projection to $\mathfrak{f}_E$ of an element of E_{0+} ; that is, that $c_\alpha = 0$. (Both inequalities of valuations in the previous sentence are actually equalities; but we do not need this.) Thus V_α would be totally B-isotropic — which, by orthogonality of the sum in (5.2.5), would be a contradiction of the non-degeneracy of B.

We claim that $t_\alpha N_{F_\alpha/F_{\pm\alpha}}(w_\alpha)^{-1}$ projects to a square in $\mathfrak{f}_\alpha^\times$ if and only if $\mathbf{G}_{\pm\alpha}$ is $F_{\pm\alpha}$-split; i.e., if and only if $\mathrm{Lie}(\mathbf{G}_{\pm\alpha})$ is $F_{\pm\alpha}$-isomorphic to $\mathfrak{sl}_2$. Once we have shown this, we will have that

$$(**) \quad \mathrm{sgn}_\mathfrak{f}(N_{\mathfrak{f}_\alpha/\mathfrak{f}}(c_\alpha)) = \mathrm{sgn}_{\mathfrak{f}_\alpha}(c_\alpha)$$
$$= \mathrm{sgn}_{\mathfrak{f}_\alpha}\big(N_{F_\alpha/F_{\pm\alpha}}(w_\alpha)d\alpha^\vee(X^*)d\alpha(Y_i)\big)\,\mathrm{sgn}_{F_{\pm\alpha}}(\mathbf{G}_{\pm\alpha}).$$

If the image in $\mathfrak{f}_\alpha$ of $t_\alpha N_{F_\alpha/F_{\pm\alpha}}(w_\alpha)^{-1}$ equals $\overline{\theta}^{-2}$ for some $\overline{\theta} \in \mathfrak{f}_\alpha^\times$, then, since $\mathfrak{f}_\alpha = \mathfrak{f}_{\pm\alpha}$ and $p \neq 2$, we may find an element $\theta \in F_{\pm\alpha}^\times$ such that $t_\alpha N_{F_\alpha/F_{\pm\alpha}}(w_\alpha)^{-1} = \theta^{-2}$. Then the unique F_α-linear map $\mathrm{Lie}(\mathbf{G}_{\pm\alpha}) \longrightarrow \mathfrak{sl}_2$ satisfying

$$X_\alpha \mapsto \frac{w_\alpha}{2\theta}\begin{pmatrix} -1 & \varpi_\alpha^{-1} \\ -\varpi_\alpha & 1 \end{pmatrix}, \quad \eta_\alpha X_\alpha \mapsto \frac{\eta_\alpha w_\alpha}{2\theta}\begin{pmatrix} -1 & -\varpi_\alpha^{-1} \\ \varpi_\alpha & 1 \end{pmatrix},$$

$$H_\alpha \mapsto \begin{pmatrix} 0 & \varpi_\alpha^{-1} \\ \varpi_\alpha & 0 \end{pmatrix}$$

is an $F_{\pm\alpha}$-isomorphism of Lie algebras.

Suppose, on the other hand, that $\iota : \mathrm{Lie}(\mathbf{G}_{\pm\alpha}) \longrightarrow \mathfrak{sl}_2$ is an $F_{\pm\alpha}$-isomorphism of Lie algebras. Then the co-character lattice of $\iota(\mathbf{t}^\alpha)$ contains a simultaneous (-1)-eigenvector for every element of $\eta_\alpha \cdot \mathrm{Gal}(F^{\mathrm{sep}}/F_\alpha)$, so the Cartan $F_{\pm\alpha}$-subalgebra $\iota(\mathbf{t}^\alpha)$ is $\mathrm{GL}_2(F_{\pm\alpha})$-conjugate to the Cartan $F_{\pm\alpha}$-subalgebra $\mathbf{t}'$ spanned by $\begin{pmatrix} 0 & \varpi_\alpha^{-1} \\ \varpi_\alpha & 0 \end{pmatrix}$. Replace ι by its composition with the indicated conjugation. Note that $d\alpha \circ \iota^{-1}$ is a weight for the adjoint action of $\mathbf{t}'$ on $\mathfrak{sl}_2$, hence is of the form $\pm d\alpha'$, where $d\alpha'$ is the functional on $\mathbf{t}'$ sending $\begin{pmatrix} 0 & \varpi_\alpha^{-1} \\ \varpi_\alpha & 0 \end{pmatrix}$ to 2. After further conjugating by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ if necessary,

we may, and do, assume that $d\alpha \circ \iota^{-1} = d\alpha'$. Then there is a constant $t'_\alpha \in F_\alpha$ such that

$$\iota(X_\alpha) = t'_\alpha \begin{pmatrix} -1 & \varpi_\alpha^{-1} \\ -\varpi_\alpha & 1 \end{pmatrix},$$

hence

$$\iota(\eta_\alpha X_\alpha) = \eta_\alpha(\iota(X_\alpha)) = \eta_\alpha(t'_\alpha) \begin{pmatrix} -1 & -\varpi_\alpha^{-1} \\ \varpi_\alpha & 1 \end{pmatrix}.$$

Recall that $\iota(H_\alpha) \in \mathbf{t}'(F_\alpha)$ is a scalar multiple of $\begin{pmatrix} 0 & \varpi_\alpha^{-1} \\ \varpi_\alpha & 0 \end{pmatrix}$. Since

$$d\alpha'(\iota(H_\alpha)) = d\alpha(H_\alpha) = 2 = d\alpha' \begin{pmatrix} 0 & \varpi_\alpha^{-1} \\ \varpi_\alpha & 0 \end{pmatrix},$$

in fact $\iota(H_\alpha) = \begin{pmatrix} 0 & \varpi_\alpha^{-1} \\ \varpi_\alpha & 0 \end{pmatrix}$. Thus,

$$\iota(t_\alpha H_\alpha) = \iota([X_\alpha, \eta_\alpha X_\alpha]) = [\iota(X_\alpha), \iota(\eta_\alpha X_\alpha)] = N_{F_\alpha/F_{\pm\alpha}}(2t'_\alpha)\iota(H_\alpha).$$

That is, $t_\alpha = N_{F_\alpha/F_{\pm\alpha}}(2t'_\alpha)$. Since $\mathrm{ord}(t_\alpha) = r - i$, we have $\mathrm{ord}(t'_\alpha) = (r-i)/2$. Then

$$t_\alpha N_{F_\alpha/F_{\pm\alpha}}(w_\alpha)^{-1} = N_{F_\alpha/F_{\pm\alpha}}(2t'_\alpha w_\alpha^{-1}) \equiv (2t'_\alpha w_\alpha^{-1})^2 \pmod{(F_\alpha^\times)_{0+}},$$

i.e., $t_\alpha N_{F_\alpha/F_{\pm\alpha}}(w_\alpha)^{-1}$ projects to the square of $\overline{2t'_\alpha w_\alpha^{-1}}$.

By Lemma 3.1, $\mathrm{sgn}_{\mathfrak{f}}(\Delta) = (-1)^{f_\alpha+1}$. Now $(*)$ and $(**)$ give

$$(5.2.19) \quad \mathrm{sgn}_{\mathfrak{f}}(\det B|_{V_\alpha})\mathfrak{G}_\Lambda(\mathfrak{f})^{\dim_{\mathfrak{f}} V_\alpha}$$
$$= (-1)^{f_\alpha+1}\left(\mathrm{sgn}_{\mathfrak{f}}(\tfrac{1}{2}e_\alpha)\mathfrak{G}_\Lambda(\mathfrak{f})\right)^{f_\alpha}$$
$$\times \mathrm{sgn}_{\mathfrak{f}_\alpha}(N_{F_\alpha/F_{\pm\alpha}}(w_\alpha)d\alpha^\vee(X^*)d\alpha(Y_i))\,\mathrm{sgn}_{F_{\pm\alpha}}(\mathbf{G}_{\pm\alpha})$$
$$= -(-\mathfrak{G}_\Lambda(\mathfrak{f}))^{f_\alpha}$$
$$\times \mathrm{sgn}_{\mathfrak{f}_\alpha}(\tfrac{1}{2}e_\alpha N_{F_\alpha/F_{\pm\alpha}}(w_\alpha)d\alpha^\vee(X^*)d\alpha(Y_i))\,\mathrm{sgn}_{F_{\pm\alpha}}(\mathbf{G}_{\pm\alpha}).$$

We have used that $\mathrm{sgn}_{\mathfrak{f}_\alpha}(n) = \mathrm{sgn}_{\mathfrak{f}}(n)^{f_\alpha}$ for $n \in \mathfrak{f}$. (Note that, if f_α is even, then $\tfrac{1}{2}e_\alpha \in \mathfrak{f}^\times \subseteq (\mathfrak{f}_\alpha^\times)^2$, and $\mathfrak{G}_\Lambda(\mathfrak{f})^{f_\alpha} = \mathrm{sgn}_{\mathfrak{f}}(-1)^{f_\alpha/2}$.)

Upon combining (†) with (5.2.17)–(5.2.19), and the facts that

- $d\alpha(Y_i) \in (\alpha(\gamma_i) - 1) + E_{i+}$ (by Lemma A.5),
- $\alpha(\gamma_i) \in \alpha((\gamma_{<r})_{\geq i}) + E_{i+}$, and
- $\alpha((\gamma_{<r})_{\geq i}) = \alpha(\gamma_{<r})$,

we obtain the desired formula. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.3. **A formula for θ_σ on G'.** The following easy technical result on integration is probably well known, but we could not find a reference.

Lemma 5.3.1. *Suppose that A is a locally compact, Hausdorff topological group, and B and C are closed subgroups of A such that*

- *$B \backslash A$ carries a quotient measure;*
- *the image of C under the projection $A \longrightarrow B \backslash A$ is open; and*
- *$B \cap C$ is compact.*

Then, for any right Haar measures da and db, on A and B, respectively, there is a right Haar measure dc on C such that

$$\int_{BC} f(a) da = \int_C \int_B f(bc) db \, dc$$

for all continuous, compactly supported functions f on A.

Proof. The choice of da and db fixes a choice of quotient measure $d\dot{a}$ on $B \backslash A$. Since $(B \cap C) \backslash C$ embeds naturally as an open subset of $B \backslash A$, $d\dot{a}$ induces a quotient measure $d\dot{c}$ on $(B \cap C) \backslash C$. Let db' be the Haar measure on $B \cap C$ such that $\operatorname{meas}_{db'}(B \cap C) = 1$. The choice of $d\dot{c}$ and db' fixes a choice of measure dc on C. Then

$$\int_{BC} f(a) da = \int_A [BC](a) f(a) da$$

$$= \int_{B \backslash A} \int_B [BC](ba) f(ba) db \, d\dot{a}$$

$$= \int_{B \backslash A} [BC](a) \int_B f(ba) db \, d\dot{a}.$$

Since the support of the outer integral is contained in (the image of) $(B \cap C) \backslash C$, we have by our choice of $d\dot{c}$ that

$$\int_{BC} f(a) da = \int_{(B \cap C) \backslash C} \int_B f(bc) db \, d\dot{c}$$

$$= \int_{(B \cap C) \backslash C} \int_{B \cap C} \int_B f(bb'c) db \, db' \, d\dot{c}$$

$$= \int_C \int_B f(bc) db \, dc. \quad \square$$

Recall that γ is semisimple and $x \in \mathcal{B}_r(\gamma)$.

Proposition 5.3.2.

$$\theta_\sigma(\gamma) = \sum_{g \in [\![\gamma;x,r]\!]_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \backslash [\![\gamma;x,r]\!]^{(s)}} \theta_{\tilde{\rho}}(^g\gamma).$$

Proof. By the Frobenius formula,

$$\theta_\sigma(\gamma) = \sum_{g \in \mathrm{stab}_{G'}(\overline{x}) G_{x,s} \setminus \mathrm{stab}_{G'}(\overline{x}) G_{x,0+}} \dot{\theta}_{\tilde{\rho}}(^g\gamma).$$

Since $\mathrm{stab}_{G'}(\overline{x}) G_{x,s} \cap G_{x,0+} = G'_{x,0+} G_{x,s}$ — which, by Proposition 5.40 of [6], is $(G', G)_{x,(0+,s)}$ — the indexing set for the sum is naturally in bijection with $(G', G)_{x,(0+,s)} \setminus G_{x,0+}$. Thus

$$(*) \qquad\qquad \theta_\sigma(\gamma) = \int_{G_{x,0+}} \dot{\theta}_{\tilde{\rho}}(^g\gamma) dg,$$

where dg is the Haar measure on $G_{x,0+}$ normalized so that $(G', G)_{x,(0+,s)}$ has measure 1.

Put $S_\infty = G'_{x,0+}[\![\gamma; x, r]\!] = \left\{ g \in G_{x,0+} \mid j^\perp(g) = \infty \right\}$, and, for $i_0, j_0 \in \mathbb{R}$, put

$$S_{i_0 j_0} = \left\{ g \in G_{x,0+} \mid i^\perp(g) = i_0, j^\perp(g) = j_0 \right\}.$$

Note that the sets $S_{i_0 j_0}$, together with S_∞, form a partition of $G_{x,0+}$. By Remark 5.1.2, they are open. We will show that the portion of $(*)$ taken over each $S_{i_0 j_0}$ vanishes, so that the integral may be taken instead over S_∞.

Fix $i_0, j_0 \in \mathbb{R}$, and put

- $t_0 = i_0 + j_0$,
- $H = C_G^{(i_0)}(\gamma)$,
- $H' = C_{G'}^{(i_0)}(\gamma)$,
- $B = (G', G)_{x,(0+,j_0+)}$, and
- $C = \left\{ g \in (H, G)_{x,(0+,t_0+)} \mid [\gamma^{-1}, g] \in (H', H)_{x,(t_0+,t_0)} \right\}$.

We claim that $S_{i_0 j_0} \subseteq BC$.

This is obvious if $S_{i_0 j_0} = \emptyset$, so assume that $S_{i_0 j_0} \neq \emptyset$. Fix $s \in S_{i_0 j_0}$, so $t^\perp(s) = t_0$ (by Corollary 5.1.4). By Remark 5.1.2 and Corollary 5.1.5, $i_0 < r - 2j_0$ and $t_0 < r - j_0$. By Lemma 5.1.6, there is

$$b_1 \in (G', G)_{x,(0+,j_0+)} = B$$

such that $[\gamma^{-1}, b_1^{-1} s] \in (H', H)_{x,(t_0+,t_0)}$. By Lemma 9.8 of [6], $b_1^{-1} s \in [\gamma; x, t_0]$, where $[\gamma; x, t_0]$ is as in Definition 9.3 of *loc. cit.* Put $s_1 = b_1^{-1} s$. By Remark 5.1.2, $i^\perp(s_1) = i_0$, $j^\perp(s_1) = j_0$, and $t^\perp(s_1) = t_0$. Thus

$$s_1 \in G'_{x,0+}[\![\gamma; x, r]\!](H, G)_{x,(j_0,j_0+)}.$$

By Proposition 5.40 and Remark 6.7(4) of [6],

$$[\![\gamma; x, r]\!](H, G)_{x,(j_0,j_0+)} = [\![\gamma; x, r]\!]^{(j_0)}(H, G)_{x,(j_0,j_0+)};$$
$$(H, G)_{x,(j_0,j_0+)} = (H', G)_{x,(j_0,j_0+)}(H', H)_{x,(j_0+,j_0)};$$

and, since the commutator of $G_{x,0+}$ with $(H', G)_{x,(j_0,j_0+)} \subseteq G_{x,j_0}$ lies in $G_{x,j_0+} \subseteq (H', G)_{x,(j_0,j_0+)}$, we have that $[\![\gamma; x, r]\!]^{(j_0)} \subseteq G_{x,0+}$ normalizes

$(H', G)_{x,(j_0, j_0+)}$. Thus we may write $s_1 = k'k_-k$, with $k' \in G'_{x,0+}(H', G)_{x,(j_0, j_0+)}$, $k_- \in \llbracket \gamma; x, r \rrbracket^{(j_0)}$, and $k \in (H', H)_{x,(j_0+, j_0)}$. By Proposition 5.40 of *loc. cit.*, $k' \in (G', G)_{x,(0+, j_0+)} = B$. By Remark 6.7(1) of *loc. cit.*,

$$(5.3.1) \quad \llbracket \gamma; x, r \rrbracket^{(j_0)} \subseteq [\gamma; x, r - j_0]^{(j_0)}$$
$$\subseteq [\gamma; x, r - j_0] \cap C_G^{(r-2j_0)}(\gamma)_{x,0+} \subseteq [\gamma; x, t_0+] \cap H_{x,0+} \,.$$

Also, $(H', H)_{x,(j_0+, j_0)} \subseteq H_{x,j_0} \subseteq [\gamma; x, t_0]$. Since $s_1 \in [\gamma; x, t_0]$, this means that $k' \in [\gamma; x, t_0] \cap (G', G)_{x,(0+, j_0+)}$. By Lemma 5.29 and Proposition 5.40 of *loc. cit.*,

$$[\gamma; x, t_0] \cap (G', G)_{x,(0+, j_0+)} = [\gamma; x, t_0]_{G'}^{(j_0+)}([\gamma; x, t_0] \cap G_{x,j_0+}).$$

Write $k' = k''k'_+$, with $k'' \in [\gamma; x, t_0]_{G'}^{(j_0+)}$ and $k'_+ \in [\gamma; x, t_0] \cap G_{x,j_0+}$. By Remark 6.7(1) of *loc. cit.*, $[\gamma; x, t_0]_{G'}^{(j_0+)} \subseteq H'_{x,0+}$, so

$$[\gamma^{-1}, k''] \in H'_{x,0+} \cap G'_{x,t_0} = H'_{x,t_0} \subseteq (H', H, G)_{x,(t_0, t_0+, t_0)}.$$

By Lemma 9.1 of *loc. cit.*,

$$[\gamma^{-1}, k'_+] \in (H, G)_{x,(t_0+, t_0)} \subseteq (H', H, G)_{x,(t_0, t_0+, t_0)}.$$

Since the commutator of $G_{x,0+}$ with $(H', H, G)_{x,(t_0, t_0+, t_0)} \subseteq G_{x,t_0}$ lies in $G_{x,t_0+} \subseteq (H', H, G)_{x,(t_0, t_0+, t_0)}$, in particular $k'' \in G_{x,0+}$ normalizes $(H', H, G)_{x,(t_0, t_0+, t_0)}$. Thus

$$[\gamma^{-1}, k'] \in (H', H, G)_{x,(t_0, t_0+, t_0)}.$$

Now, using (5.3.1) and imitating the above argument that $[\gamma^{-1}, k''] \in H'_{x,t_0}$, we see that $[\gamma^{-1}, k_-] \in H_{x,t_0+} \subseteq (H', H)_{x,(t_0+, t_0)}$. Also, by Lemma 5.32 (or Corollary 5.21, if $i_0 = 0$) of [6],

$$[\gamma^{-1}, k] = [\gamma_{\geq i_0}^{-1}, k] \in (H', H)_{x,((i_0+j_0)+, i_0+j_0)} = (H', H)_{x,(t_0+, t_0)}.$$

Thus, since $k_- \in H_{x,0+}$ normalizes $(H', H)_{x,(t_0+, t_0)}$,

$$[\gamma^{-1}, k_-k] \in (H', H)_{x,(t_0+, t_0)}.$$

Thus $s = bc$, where $b := b_1 k' \in B$ and $c := k_-k \in C$.

Now we claim that

$$\int_{S_{i_0 j_0}} \dot\theta_{\tilde\rho}(^g\gamma)dg = 0.$$

Once again, this is obvious if $S_{i_0 j_0} = \emptyset$, so suppose $S_{i_0 j_0} \neq \emptyset$. Note that $(H', H)_{x, ((r-t_0)+, r-t_0)} \subseteq B$. By Lemma 5.3.1, applied to the function sending $g \in G$ to $[S_{i_0 j_0}](g) \dot{\theta}_{\tilde{\rho}}(^g \gamma)$, we have that

$$
(5.3.2) \quad \int_{S_{i_0 j_0}} \dot{\theta}_{\tilde{\rho}}(^g \gamma) dg = (\text{const}) \int_B \int_C [S_{i_0 j_0}](bc) \dot{\theta}_{\tilde{\rho}}(^{bc} \gamma) dc\, db
$$

$$
= (\text{const}) \int_B \int_C \int_{(H', H)_{x, ((r-t_0)+, r-t_0)}} [S_{i_0 j_0}](bhc) \dot{\theta}_{\tilde{\rho}}(^{bhc} \gamma) dh\, dc\, db.
$$

Suppose $b \in B$, $h \in (H', H)_{x, ((r-t_0)+, r-t_0)}$, and $c \in C$. By Remark 5.1.2, $[S_{i_0 j_0}](bhc) = [S_{i_0 j_0}](c)$. We have that

$$
(5.3.3) \qquad\qquad {}^{hc}\gamma = [h, \gamma] \cdot {}^c\gamma \cdot \big[[c, \gamma^{-1}], h\big].
$$

By Lemma 5.32 (or Corollary 5.21, if $i_0 = 0$) of [6],
$$
(5.3.4)
$$
$$
[h, \gamma] = [h, \gamma_{\geq i_0}] \in (H', H)_{x, (r-t_0+i_0)+, r-t_0+i_0)} \subseteq (G', G)_{x, (r+, r-j_0)}.
$$

By the definition of the group C, we have $[c, \gamma^{-1}] \in G_{x, t_0}$. Thus, since $h \in G_{x, r-t_0}$, we have

$$
(5.3.5) \qquad\qquad \big[[c, \gamma^{-1}], h\big] \in G_{x, r}.
$$

Combining (5.3.3)–(5.3.5) gives

$$
{}^{bhc}\gamma \in {}^b(G', G)_{x, (r+, r-j_0)} \cdot {}^{bc}\gamma \cdot {}^b\big[[c, \gamma^{-1}], h\big]
$$
$$
\subseteq (G', G)_{x, (r+, r-j_0)} \cdot {}^{bc}\gamma \cdot \big[[c, \gamma^{-1}], h\big] G_{x, r+}
$$

for $b \in B = (G', G)_{x, (0+, j_0+)}$. The containment on the second line follows from the fact that, by Corollary 5.18 of [6], $(G', G)_{x, (r+, r-j_0)}$ is normalized by $B = (G', G)_{x, (0+, j_0)}$. Now, by Lemma 2.5 and the fact that $(G', G)_{x, (r+, r-j_0)} \subseteq \ker \hat{\phi}$,

$$
\dot{\theta}_{\tilde{\rho}}(^{bhc}\gamma) = \hat{\phi}\big(\big[[c, \gamma^{-1}], h\big]\big) \dot{\theta}_{\tilde{\rho}}(^{bc}\gamma) = \big[[\gamma^{-1}, c], \hat{\phi}\big](h) \dot{\theta}_{\tilde{\rho}}(^{bc}\gamma),
$$

where $\big[[\gamma^{-1}, c], \hat{\phi}\big]$ is the character of $G_{x, r-t_0}$ given by $g \mapsto \hat{\phi}\big(\big[[c, \gamma^{-1}], g\big]\big)$.

In particular, the inner integral in (5.3.2) is 0 unless $c \in C \cap S_{i_0 j_0}$ and $\big[[\gamma^{-1}, c], \hat{\phi}\big]$ is trivial on $(H', H)_{x, ((r-t_0)+, r-t_0)}$.

Fix $c \in C \cap S_{i_0 j_0}$ for which the indicated character is trivial. If $g \in G'_{x, r-t_0}$, then, by two applications of Hypothesis A.1(5)

$$
\big[[\gamma^{-1}, c], \hat{\phi}\big](g) = \hat{\phi}\big(\big[[c, \gamma^{-1}], g\big]\big) = 1.
$$

If $g \in (H, G)_{x, ((r-t_0)+, r-t_0)}$, then, since $[c, \gamma^{-1}] \in (H', H)_{x, (t_0+, t_0)} \subseteq H_{x, t_0}$, we have by Lemma 5.32 of [6] that $\big[[c, \gamma^{-1}], g\big] \in (H, G)_{x, (r+, r)}$. By Lemma 4.1 and Remark 4.2, $(H, G)_{x, (r+, r)} \subseteq \ker \hat{\phi}$, so $g \in \ker \big[[\gamma^{-1}, c], \hat{\phi}\big]$.

We have seen that $\big[[\gamma^{-1}, c], \hat{\phi}\big]$ is trivial on the group generated by $(H', H)_{x,((r-t_0)+,r-t_0)}$, $G'_{x,r-t_0}$, and $(H, G)_{x,((r-t_0)+,r-t_0)}$, which, by Proposition 5.40 of [6], is all of $G_{x,r-t_0}$. By Lemma B.1, this means that $[\gamma^{-1}, c] \in (G', G)_{x,(0+,t_0+)}$. Since $c \in C$, also $[\gamma^{-1}, c] \in (G', G)_{x,(t_0+,t_0)}$; so in fact $[\gamma^{-1}, c] \in G_{x,t_0+}$. This contradicts the fact that $t_0 = t^{\perp}(c)$. Thus the inner integral in (5.3.2) is always 0, so $\int_{S_{i_0 j_0}} \dot{\theta}_{\tilde{\rho}}(^g\gamma)dg = 0$, as desired.

By Remarks 6.7(1) and 6.7(4) of [6], $G_{x,s}[\![\gamma; x, r]\!]^{(s)} = [\![\gamma; x, r]\!]$; and, by Proposition 5.40 of *loc. cit.*, $G'_{x,0+}G_{x,s} = (G', G)_{x,(0+,s)}$; so $G'_{x,0+}[\![\gamma; x, r]\!] = (G', G)_{x,(0+,s)}[\![\gamma; x, r]\!]^{(s)}$. By Lemma 5.3.1, there is a measure dh on $[\![\gamma; x, r]\!]^{(s)}$ such that

$$
\int_{G'_{x,0+}[\![\gamma;x,r]\!]} f(g)dg = \int_{[\![\gamma;x,r]\!]^{(s)}} \int_{(G',G)_{x,(0+,s)}} f(gh)dg\,dh
$$

for all continuous functions f on $G_{x,0+}$. By definition, $\mathrm{meas}_{dg}((G', G)_{x,(0+,s)}) = 1$, so

$$
\begin{aligned}
\mathrm{meas}_{dh}([\![\gamma; x, r]\!]^{(s)}) &= \mathrm{meas}_{dg}(G'_{x,0+}[\![\gamma; x, r]\!]) \\
&= \Big[G'_{x,0+}[\![\gamma; x, r]\!] : (G', G)_{x,(0+,s)}\Big] \\
&= \Big[[\![\gamma; x, r]\!]^{(s)} : [\![\gamma; x, r]\!]^{(s)} \cap (G', G)_{x,(0+,s)}\Big].
\end{aligned}
$$

By Lemma 5.29 of [6], $[\![\gamma; x, r]\!]^{(s)} \cap (G', G)_{x,(0+,s)} = [\![\gamma; x, r]\!]^{(s)}_{G'} C_G^{(0+)}(\gamma)_{x,s}$. Thus

$$
\begin{aligned}
\theta_\sigma(\gamma) &= \int_{G'_{x,0+}[\![\gamma;x,r]\!]} \dot{\theta}_{\tilde{\rho}}(^g\gamma)dg = \int_{[\![\gamma;x,r]\!]^{(s)}} \int_{(G',G)_{x,(0+,s)}} \dot{\theta}_{\tilde{\rho}}(^{gh}\gamma)dg\,dh \\
&= \int_{[\![\gamma;x,r]\!]^{(s)}} \dot{\theta}_{\tilde{\rho}}(^h\gamma)dh = \sum_{g \in [\![\gamma;x,r]\!]^{(s)}_{G'} C_G^{(0+)}(\gamma)_{x,s} \backslash [\![\gamma;x,r]\!]^{(s)}} \dot{\theta}_{\tilde{\rho}}(^g\gamma). \quad \square
\end{aligned}
$$

Proposition 5.3.3.

$$
\begin{aligned}
\theta_\sigma(\gamma) = {}& \Big[[\![\gamma_{<r}; x, r]\!] : [\![\gamma_{<r}; x, r]\!]_{G'}G_{x,s}\Big]^{1/2} \\
&\times \Big[[\![\gamma_{<r}; x, r+]\!] : [\![\gamma_{<r}; x, r+]\!]_{G'}G_{x,s+}\Big]^{1/2} \\
&\times \mathfrak{G}(\phi, \gamma_{<r})\varepsilon(\phi, \gamma_{<r})\theta_{\tau_{d-1}}(\gamma),
\end{aligned}
$$

where $\mathfrak{G}(\phi, \gamma_{<r})$ is as in Proposition 5.2.13 and $\varepsilon(\phi, \gamma_{<r})$ is as in Proposition 3.8.

Proof. By Lemma 2.5, applied to $\sigma = \sigma_d$ and τ_{d-1} (using the fact that $G'_{x,r_{d-1}} \subseteq G'_{x,r_{d-2}+}$), it suffices to verify the desired equality when $\gamma = \gamma_{<r}$.

Then, by Proposition 5.3.2 and Lemma 2.5, we have

$$
\theta_\sigma(\gamma) = \sum_{g \in [\![\gamma;x,r]\!]_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \backslash [\![\gamma;x,r]\!]^{(s)}} \dot{\theta}_{\tilde{\rho}}({}^g\gamma)
$$

$$
= \theta_{\tilde{\rho}}(\gamma) \sum_{g \in [\![\gamma;x,r]\!]_{G'}^{(s)} C_G^{(0+)}(\gamma)_{x,s} \backslash [\![\gamma;x,r]\!]^{(s)}} \hat{\phi}([\gamma^{-1}, g]) = \theta_{\tilde{\rho}}(\gamma) \, |\widetilde{\mathfrak{G}}| \, \mathfrak{G}
$$

(where $\widetilde{\mathfrak{G}} = \widetilde{\mathfrak{G}}(\phi, \gamma)$ and $\mathfrak{G} = \mathfrak{G}(\phi, \gamma)$ are the quantities calculated in §5.2). If $\gamma \in {}^{\mathrm{stab}_{G'}(\overline{x})} K^{d-1}$ — say, $\gamma = {}^g k$, with $g \in \mathrm{stab}_{G'}(\overline{x})$ and $k \in K^{d-1}$ — then Lemma 2.4 and Proposition 3.8 show that

$$
\theta_{\tilde{\rho}}(\gamma) = \theta_{\tilde{\rho}}(k) = \theta_{\tilde{\phi}}(k \ltimes 1)\theta_{\tau_{d-1}}(k) = \theta_{\tilde{\phi}}(\gamma \ltimes 1)\theta_{\tau_{d-1}}(\gamma)
$$

$$
= |(C_{G'}^{(0+)}(\gamma), C_G^{(0+)}(\gamma))_{x,(r,s):(r,s+)}|^{1/2} \, \varepsilon(\phi, \gamma)\theta_{\tau_{d-1}}(\gamma).
$$

As in the proof of Lemma 2.5, we see that $K \cap \mathrm{stab}_{G'}(\overline{x}) = K^{d-1}$. Thus, since $J \subseteq K$, we have ${}^{\mathrm{stab}_{G'}(\overline{x})J} K \cap \mathrm{stab}_{G'}(\overline{x}) = {}^{\mathrm{stab}_{G'}(\overline{x})}(K \cap \mathrm{stab}_{G'}(\overline{x})) = {}^{\mathrm{stab}_{G'}(\overline{x})} K^{d-1}$. Since $\gamma \in \mathrm{stab}_{G'}(\overline{x})$, we have that, if $\gamma \notin {}^{\mathrm{stab}_{G'}(\overline{x})} K^{d-1}$ — so that $\theta_{\tau_{d-1}}(\gamma) = 0$ — then $\gamma \notin {}^{\mathrm{stab}_{G'}(\overline{x})J} K$ — so that again

$$
\theta_{\tilde{\rho}}(\gamma) = 0 = |(C_{G'}^{(0+)}(\gamma), C_G^{(0+)}(\gamma))_{x,(r,s):(r,s+)}|^{1/2} \, \varepsilon(\phi, \gamma)\theta_{\tau_{d-1}}(\gamma).
$$

To complete the proof, we note that, by Proposition 5.2.12,

$$
|\widetilde{\mathfrak{G}}| \cdot |(C_{G'}^{(0+)}(\gamma), C_G^{(0+)}(\gamma))_{x,(r,s):(r,s+)}|^{1/2}
$$

equals

$$
\left[[\![\gamma;x,r]\!] : [\![\gamma;x,r]\!]_{G'} G_{x,s} \right]^{1/2} \left[[\![\gamma;x,r+]\!] : [\![\gamma;x,r+]\!]_{G'} G_{x,s+} \right]^{1/2}. \qquad \square
$$

6. INDUCTION TO G

In this section, we compute the character of the representation $\tau = \tau_d$ of $\mathrm{stab}_G(\overline{x})$ induced from the representation (σ, K_σ) whose character we computed in §5. If $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic, then we also compute the character of the representation $\pi = \pi_d$ of G induced from (σ, K_σ). As in §5, unless certain tameness and compactness hypotheses are satisfied, we must place mild restrictions on the elements that we consider.

Namely, we fix throughout this section an element $\gamma \in G$, and assume that γ has a normal r-approximation; but, unless otherwise stated, we do *not* assume that $\gamma \in G'$ or $x \in \mathcal{B}_r(\gamma)$. By Lemma 8.1 of [6], under suitable assumptions on $\mathbf{G}$, any bounded-modulo-$Z(G)$ element of G that belongs to a tame F-torus will do. By [13] or [24], $\Theta_\pi(\gamma) = 0$ unless γ is bounded modulo $Z(G)$, and the domain of τ is already bounded modulo $Z(G)$; so, under these assumptions, we need only require that γ be tame. (Remember

that an element or subgroup of G is said to be bounded modulo $Z(G)$ if its orbits in $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, F)$ are bounded in the sense of metric spaces.)

Lemma 6.1. *If* $\mathbf{M}$ *is a Levi* F*-subgroup of* $\mathbf{G}$ *and* $\delta \in G^{\mathrm{ss}}$*, then*

$$M \backslash \big\{ g \in G \,\big|\, {}^g\delta \in M \big\} / C_G(\delta)^\circ$$

is finite.

Note that $\mathbf{M}$ above need not be an F-Levi subgroup (i.e., a Levi component of a parabolic F-subgroup).

Proof. Put $\mathbf{H} = C_\mathbf{G}(\delta)$ and $\mathcal{C} = \big\{ g \in G \,\big|\, {}^g\delta \in M \big\}$. Since every $(N_G(M), H^\circ)$-double coset is a finite union of (M, H°)-double cosets, it suffices to show that $N_G(M)\backslash\mathcal{C}/H^\circ$ is finite.

Let $\mathbf{S}$ be a maximal torus in $\mathbf{G}$ containing δ. For $g \in \mathcal{C}$, we have that $Z(\mathbf{M}^g)^\circ \subseteq \mathbf{S} \subseteq \mathbf{H}^\circ$, so $\mathbf{M}^g \cap \mathbf{H}^\circ = C_{\mathbf{H}^\circ}(Z(\mathbf{M}^g)^\circ)$ is a Levi subgroup (necessarily defined over F) of $\mathbf{H}^\circ$. Consider the H-equivariant map f from $N_G(M)\backslash\mathcal{C}$ to the set of Levi F-subgroups of $\mathbf{H}^\circ$ that sends $N_G(M)g$ to $\mathbf{M}^g \cap \mathbf{H}^\circ$. We claim that f is finite-to-one. Indeed, for $g_0 \in \mathcal{C}$, fix a torus $\mathbf{T}^{g_0}$ that is maximal in $\mathbf{M}^{g_0} \cap \mathbf{H}^\circ$, hence in $\mathbf{G}$. Then $N_G(M)g \mapsto \mathbf{M}^g$ is an injection from the fiber of f over $\mathbf{M}^{g_0} \cap \mathbf{H}^\circ$ into the set of Levi subgroups of $\mathbf{G}$ containing $\mathbf{T}^{g_0}$, which is finite.

Thus there is a finite-to-one map from $N_G(M)\backslash\mathcal{C}/H^\circ$ to the set of H°-orbits of Levi F-subgroups of $\mathbf{H}^\circ$. Recall that there are only finitely many $\mathbf{H}^\circ(F^{\mathrm{sep}})$-orbits of Levi subgroups of $\mathbf{H}^\circ$. Thus it suffices to show that every such orbit contains at most finitely many H°-orbits of Levi F-subgroups.

Accordingly, fix a Levi subgroup $\mathbf{L} \subseteq \mathbf{H}^\circ$. Clearly, it suffices to consider the case where $\mathbf{L}$ is F-rational. Then the intersection of the $\mathbf{H}^\circ(F^{\mathrm{sep}})$-orbit of $\mathbf{L}$ with the set of Levi F-subgroups of $\mathbf{H}^\circ$ is

$$\big\{ \mathbf{L}^h \,\big|\, h \in \mathbf{H}^\circ(F^{\mathrm{sep}}) \text{ and } h\sigma(h)^{-1} \in N_{\mathbf{H}^\circ}(\mathbf{L})(F^{\mathrm{sep}}) \text{ for } \sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F) \big\},$$

which is naturally in H°-equivariant bijection with $(N_{\mathbf{H}^\circ}(\mathbf{L})\backslash\mathbf{H}^\circ)(F)$. Thus, it suffices to show that $(\mathbf{L}\backslash\mathbf{H}^\circ)(F)/H^\circ$ is finite. Standard Galois cohomology arguments show that this latter set is in bijection with the kernel of the natural map $H^1(F^{\mathrm{sep}}/F, \mathbf{L}(F^{\mathrm{sep}})) \longrightarrow H^1(F^{\mathrm{sep}}/F, \mathbf{H}^\circ(F^{\mathrm{sep}}))$. Under the assumption that F has characteristic 0 and that $\mathbf{G}$ is F^{un}-split and F-quasisplit, [22, §§2.2–2.3], describes a bijection of $H^1(F^{\mathrm{un}}/F, \mathbf{L}(F^{\mathrm{un}}))$ with the set of torsion points in a certain finite quotient of the lattice of cocharacters of a certain torus (see Corollary 2.3.3 of *loc. cit.*). However, it is observed there that the splitness and quasisplitness assumptions are unnecessary (although we need to take the torus $\mathbf{T}$ of §2.3 of *loc. cit.* to be the centralizer of a maximal F^{un}-split torus containing a maximal F-split torus); and it can be checked that the proof also does not require $\mathrm{char}\, F = 0$. Thus, $H^1(F^{\mathrm{un}}/F, \mathbf{L}(F^{\mathrm{un}}))$ is finite. Since $H^1(F^{\mathrm{sep}}/F^{\mathrm{un}}, \mathbf{L}(F^{\mathrm{sep}})) = \{0\}$

(as observed in [6, §3.2]), we have by [58, §I.5.8(a)] that $H^1(F^{\text{sep}}/F, \mathbf{L}(F^{\text{sep}}))$, hence *a fortiori* the desired kernel, is also finite. $\qquad\square$

For the remainder of this paper, we fix a normal r-approximation to γ (hence to all of its conjugates and truncations), so that $\gamma_{<r}$ is a well de-fined element. For the remainder of this section, we put $\mathbf{H} = C_{\mathbf{G}}(\gamma_{<r})$. (Note that Proposition 8.4 of [6] guarantees only that $\mathbf{H}^\circ$, not necessarily $\mathbf{H}$ itself, is determined by γ; but, since we have chosen a specific normal r-approximation, there is no ambiguity.)

We need to prove a result analogous to Lemmata 10.0.5 and 10.0.6 of [22]. First, we prove an analogue of Lemma 7.0.9 of *loc. cit.*

Lemma 6.2. *Suppose that γ is regular semisimple in $\mathbf{G}$. Then $\gamma_{\geq r}$ is regular semisimple in $\mathbf{H}$.*

Proof. By Definition 6.8 and Lemma 6.13 of [6], $\gamma \in C_G^{(r)}(\gamma) = H^\circ$. Thus, there is a torus containing both γ and $\gamma_{<r}$, hence also $\gamma_{\geq r}$. In particular, $\gamma_{\geq r} \in H^\circ$ is semisimple, so it suffices to show that $C_{\mathbf{H}}(\gamma_{\geq r})^\circ$ is contained in a torus. We have that $C_{\mathbf{H}}(\gamma_{\geq r}) \subseteq C_{\mathbf{G}}(\gamma_{<r}\gamma_{\geq r}) = C_{\mathbf{G}}(\gamma)$, so $C_{\mathbf{H}}(\gamma_{\geq r})^\circ \subseteq C_{\mathbf{G}}(\gamma)^\circ$. By regularity of γ, we have that $C_{\mathbf{G}}(\gamma)^\circ$ is a torus. The proof is complete. $\qquad\square$

Lemma 6.3. *Suppose that $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic and γ is regular semisimple. If $\mathcal{K}_H$ is a compact open subgroup of H°, then*

$$g \mapsto \int_{\mathcal{K}_H} \dot\theta_\sigma(^{gk}\gamma) dk$$

is compactly supported on $G/Z(G)$.

Proof. By Lemma 6.1 (with $\mathbf{M} = \mathbf{G}'$), since K_σ contains G', the set of (K_σ, H°)-double cosets in G containing an element g with $^g\gamma_{<r} \in G'$ is finite. By Corollary 4.5, the support of the function occurring in the state-ment is contained in the union of such double cosets. Thus, it suffices to show that the restriction of the indicated function to any (K_σ, H°)-double coset has compact support.

Fix a double coset $K_\sigma g H^\circ$ in G. Since $\dot\theta_\sigma$ is invariant under conjugation by the compact-modulo-$Z(G)$ group K_σ, it suffices to show that

$$h \mapsto \int_{\mathcal{K}_H} \dot\theta_\sigma(^{ghk}\gamma) dk$$

is compactly supported on H°, modulo $Z(G)$. Suppose that $h \in H^\circ$ and $k \in \mathcal{K}_H \subseteq H^\circ$. Then $^{ghk}\gamma = (^g\gamma_{<r})(^{ghk}\gamma_{\geq r})$. Therefore, by Corollary 4.6,

$$\dot\theta_\sigma(^{ghk}\gamma) = [G_{x,0+} : G'](^g\gamma_{<r})\dot\theta_\sigma(^g\gamma_{<r}) \cdot [\mathcal{B}(\mathbf{H}, F)]((ghk)^{-1}x)$$

$$\times [G_{x,r}](^{ghk}\gamma_{\geq r})\hat\phi(^{ghk}\gamma_{\geq r}).$$

Since ${}^{hk}\gamma_{\geq r} \in H^\circ$, we have that ${}^{ghk}\gamma_{\geq r} \in G_{x,r}$ if and only if ${}^{hk}\gamma_{\geq r} \in H^\circ \cap G_{g^{-1}x,r}$. Thus it suffices to show that

$$(*) \quad h \mapsto \int_{\mathcal{K}_H} [\mathcal{B}(\mathbf{H}, F)]((ghk)^{-1}x) \cdot [H^\circ \cap G_{g^{-1}x,r}]({}^{hk}\gamma_{\geq r})\hat{\phi}^g({}^{hk}\gamma_{\geq r})dk$$

is compactly supported on H°, modulo $Z(G)$, whenever ${}^g\gamma_{<r} \in {}^{G_{x,0+}}G'$. Since $(*)$ does not change if we replace g by an element of $K_\sigma g$, we need only consider the case that ${}^g\gamma_{<r} \in G'$.

If $g^{-1}x \notin \mathcal{B}(\mathbf{H}, F)$, then the function $(*)$ vanishes. Suppose that $g^{-1}x \in \mathcal{B}(\mathbf{H}, F)$ (as well as ${}^g\gamma_{<r} \in G'$). Then, by Remark 6.10(2) of [6], ${}^g\gamma_{<r} \in \mathrm{stab}_G(\overline{x})$. Let $\Sigma_H = (\vec{\mathbf{H}}, \vec{\phi}_H, \vec{r}_H, x_H, \rho'_{H,0})$ be a cuspidal datum (see Definition 2.1) such that

    (1) $\vec{\mathbf{H}} = (\mathbf{H}^0 \subseteq \mathbf{H}^1)$, where $\mathbf{H}^0 = \mathbf{H}^\circ \cap {}^{g^{-1}}\mathbf{G}'$ and $\mathbf{H}^1 = \mathbf{H}^\circ$;
    (2) $\vec{\phi}_H = (\phi_{H,0}, 1)$, where $\phi_{H,0} = \phi^g\big|_{H^0}$;
    (3) $\vec{r}_H = (r, r)$; and
    (4) $x_H = g^{-1}x$.

(Note that $\mathbf{H}^0$, $\phi_{H,0}$ and x_H all depend on g as well as on H.) As in §2, there are associated to the datum Σ_H a compact-modulo-$Z(H)$ open subgroup K_{Σ_H} of H and a representation ρ'_{Σ_H} of K_H such that $\pi_{\Sigma_H} = \mathrm{Ind}_{K_{\Sigma_H}}^H \rho'_{\Sigma_H}$ is an irreducible supercuspidal representation of H. Put $K_{\sigma_H} = K_{\Sigma_H} H_{x,0+}$ and $\sigma_H = \mathrm{Ind}_{K_{\Sigma_H}}^{K_{\sigma_H}} \rho'_{\Sigma_H}$. Now we are in the situation of §5 (with (G, σ) there replaced by (H°, σ_H)).

By Corollary 4.6, for $h \in H^\circ$ and $k \in \mathcal{K}_H$, we have

$$\dot{\theta}_{\sigma_H}({}^{hk}\gamma_{\geq r}) = [\mathcal{B}(\mathbf{H}, F)]((hk)^{-1}x_H) \cdot [H_{x_H,r}]({}^{hk}\gamma_{\geq r})\hat{\phi}_{H,0}({}^{hk}\gamma_{\geq r})$$
$$= [\mathcal{B}(\mathbf{H}, F)]((ghk)^{-1}x) \cdot [G_{x,r}]({}^{ghk}\gamma_{\geq r})\hat{\phi}^g({}^{hk}\gamma_{\geq r}).$$

Since $\dot{\theta}_{\sigma_H}$ is a sum of matrix coefficients of the supercuspidal representation π_H, it is a cusp form (or 'supercusp form', in the language of [28, §I.3]) on H°. In particular, by Lemma 23 of [28] (the proof of which does not depend on $\mathrm{char}\, F$ being 0) and our Lemma 6.2, $(*)$ is compactly supported on H°, modulo $Z(H^\circ)$. Since $\gamma_{<r} \in G'$, we have that $Z(H^\circ) = Z(C_G(\gamma_{<r})^\circ) \subseteq G'$ is compact modulo $Z(G)$, so $(*)$ is also compactly supported on H°, modulo $Z(G)$. $\qquad\square$

The portion of the following result concerning Θ_π is the analogue of Lemma 10.0.4 of [22].

Theorem 6.4. *If $x \in \mathcal{B}_r(\gamma)$, then*

$$(6.1) \qquad \theta_\tau(\gamma) = \phi_d(\gamma) \sum_g \theta_\sigma({}^g\gamma_{<r})\hat{\mu}_{X^*}^{\mathrm{stab}_{gH}(\overline{x})}(\mathrm{e}_x^{-1}({}^g\gamma_{\geq r})).$$

If $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic and $\gamma \in G$ is regular semisimple, then

$$(6.2) \qquad \Theta_\pi(\gamma) = \phi_d(\gamma) \sum_g \theta_\sigma({}^g\gamma_{<r}) \hat{\mu}_{X^*}^{{}^gH}(\mathsf{e}_x^{-1}({}^g\gamma_{\geq r})).$$

The sums run over those double cosets in $\operatorname{stab}_{G'}(\overline{x}) G_{x,0+} \backslash \operatorname{stab}_G(\overline{x}) / \operatorname{stab}_H(\overline{x})$ or $G' G_{x,0+} \backslash G / H$, respectively, containing an element g such that ${}^g\gamma_{<r} \in G'$ and $x \in \mathcal{B}_r({}^g\gamma)$.

Here, $\hat{\mu}_{X^*}^{\operatorname{stab}_{{}^gH}(\overline{x})}$ *is the function representing the distribution* (6.4) *below, and both it and $\hat{\mu}_{X^*}^{{}^gH}$ are defined with respect to the Haar measure on ${}^gH/Z(\mathbf{G})$ normalized so that* $\operatorname{meas}(K_\sigma \cap {}^gH/Z(\mathbf{G})) = 1$.

If we used a suitable exponential map in place of e_x (one that, among other things, was conjugation invariant and defined on all the filtration lattices $\mathfrak{g}_{y,r}$ for $y \in \mathcal{B}(\mathbf{G}, F)$), then the sums in (6.1) and (6.2) could be extended over all double cosets containing an element g such that ${}^g\gamma_{<r} \in G'$, since Lemma B.4 shows that the extra summands would vanish. However, it is more convenient for our purposes to restrict the sum, so that we do not have to assume the existence of a suitable exponential map, and so that we can apply Proposition 5.3.3 (which is subject to the assumptions in force through all of §5, including that $\gamma_{\geq r} \in G_{x,r}$).

Note that the orbital integrals appearing in (6.2) are taken over the F-rational points of possibly *disconnected* groups ${}^gH$. By Lemma B.2, it is easy to describe them as sums of orbital integrals over the connected groups ${}^gH^\circ$ if necessary.

Proof. Recall that $K_\sigma = \operatorname{stab}_{G'}(\overline{x}) G_{x,0+}$, so $K_\sigma = G' G_{x,0+}$ if $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic.

First, we compute $\theta_\tau(\gamma)$ (in case $x \in \mathcal{B}_r(\gamma)$). By the Frobenius formula, we have that $\theta_\tau(\gamma) = \phi_d(\gamma) \sum_{g \in K_\sigma \backslash \operatorname{stab}_G(\overline{x})} \dot{\theta}_\sigma({}^g\gamma)$, so

$$(6.3) \quad \theta_\tau(\gamma) = \phi_d(\gamma) \sum_{g \in K_\sigma \backslash \operatorname{stab}_G(\overline{x}) / \operatorname{stab}_H(\overline{x})} \;\; \sum_{g' \in K_\sigma \backslash K_\sigma g \operatorname{stab}_H(\overline{x})} \dot{\theta}_\sigma({}^{g'}\gamma)$$

$$= \phi_d(\gamma) \sum_g \;\; \sum_{h' \in (\operatorname{stab}_{{}^gH}(\overline{x}) \cap K_\sigma) \backslash \operatorname{stab}_{{}^gH}(\overline{x})} \dot{\theta}_\sigma({}^{h'g}\gamma)$$

$$= \phi_d(\gamma) \sum_g \dot{\theta}_\sigma({}^g\gamma_{<r}) \sum_{h'} \hat{\phi}({}^{h'g}\gamma_{\geq r})$$

(where the equality on the last line follows from Lemma 2.5, and the fact that $({}^{h'g}\gamma)_{<r} = {}^g\gamma_{<r}$ for all g and h' as above). An easy formal calculation, using Hypothesis A.7(2) and the fact that $\operatorname{meas}(K_\sigma \cap \operatorname{stab}_{{}^gH}(\overline{x})/Z(\mathbf{G})) =$

$\mathrm{meas}(K_\sigma \cap {}^g H/Z(G)) = 1$, shows that

$$(*) \qquad \sum_{h' \in (\mathrm{stab}_{{}^g H}(\overline{x}) \cap K_\sigma) \setminus \mathrm{stab}_{{}^g H}(\overline{x})} \hat{\phi}({}^{h'g}\gamma_{\geq r}) = \sum_{h'} \Lambda(X^*({}^{h'}\mathsf{e}_x^{-1}({}^g\gamma_{\geq r})))$$
$$= \hat{\mu}_{X^*}^{\mathrm{stab}_{{}^g H}(\overline{x})}(\mathsf{e}_x^{-1}({}^g\gamma_{\geq r})),$$

where $\hat{\mu}_{X^*}^{\mathrm{stab}_{{}^g H}(\overline{x})}$ is the function representing the distribution

$$(6.4) \qquad\qquad f \mapsto \int_{\mathrm{stab}_{{}^g H}(\overline{x})/Z(G)} \hat{f}(\mathrm{Ad}^*(h')^{-1} X^*) d\dot{h}'$$

on ${}^g\mathfrak{h}^*$. Now fix a double coset $K_\sigma g\,\mathrm{stab}_H(\overline{x})$ with $g \in \mathrm{stab}_G(\overline{x})$. If there is no element g' in the double coset such that ${}^{g'}\gamma_{<r} \in G'$, then, by Corollary 4.4, we have that $\dot{\theta}_\sigma({}^g\gamma_{<r}) = 0$, so the summand corresponding to g on the last line of (6.3) vanishes. If ${}^g\gamma_{<r} \in G'$, then, since ${}^g\gamma_{<r} \in \mathrm{stab}_G(\overline{x})$, we have ${}^g\gamma_{<r} \in \mathrm{stab}_{G'}(\overline{x})$. In particular, ${}^g\gamma_{<r}$ is in the domain of σ, so $\dot{\theta}_\sigma({}^g\gamma_{<r}) = \theta_\sigma({}^g\gamma_{<r})$, and the summands in (6.1) and, by $(*)$, the last line of (6.3) corresponding to g are the same. Note that $gx \in \mathcal{B}_r({}^g\gamma)$, so, since $\overline{x} = g\overline{x}$, also $x \in \mathcal{B}_r({}^g\gamma)$.

Next, we compute $\Theta_\pi(\gamma)$ (in case $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic). By Harish-Chandra's integral formula, for any compact open subgroup $\mathcal{K}$ of G,

$$(6.5) \qquad \Theta_\pi(\gamma) = \frac{\deg(\pi)}{\deg(\sigma)} \phi_d(\gamma) \int_{G/Z(G)} \int_{\mathcal{K}} \dot{\theta}_\sigma({}^{g'c}\gamma) dc\, d\dot{g}',$$

where $d\dot{g}'$ is a Haar measure on $G/Z(G)$, and dc is the Haar measure on $\mathcal{K}$ normalized so that $\mathrm{meas}(\mathcal{K}) = 1$. (In characteristic 0, this was proven for supercuspidal representations — in particular, for π — by Harish-Chandra in [28]. In [51], Rader and Silberger demonstrated an analogue of this result for discrete series representations. In [4, Appendix B], Prasad provided a characteristic-free proof of a submersion principle of Harish-Chandra. Since the proof of the integral formula for characters (Theorem 12 of [28, p. 60]) relies only on the submersion principle and Lemma 23 of [28, p. 59], and since the proof of the latter does not depend on $\mathrm{char}\, F$ being 0, the correctness of the integral formula in any characteristic follows.)

For the remainder of the proof, we will assume that $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic. We claim that the inner integral in (6.5) may be replaced by an integral over $\mathcal{K}_H := \mathcal{K} \cap H^\circ$. (This is the analogue of Lemma 10.0.7 of

[22].) Indeed,

$$\frac{\deg(\sigma)}{\deg(\pi)}\phi_d(\gamma)^{-1}\Theta_\pi(\gamma) = \int_{G/Z(G)}\int_{\mathcal{K}}\dot\theta_\sigma(^{g'c}\gamma)dc\,d\dot g'$$

$$= \int_{G/Z(G)}\int_{\mathcal{K}_H}\int_{\mathcal{K}}\dot\theta_\sigma(^{g'ck}\gamma)dc\,dk\,d\dot g'$$

$$= \int_{\mathcal{K}}\int_{G/Z(G)}\int_{\mathcal{K}_H}\dot\theta_\sigma(^{g'ck}\gamma)dk\,d\dot g'\,dc$$

$$= \int_{G/Z(G)}\int_{\mathcal{K}_H}\dot\theta_\sigma(^{g'k}\gamma)dk\,d\dot g',$$

where dk is the Haar measure on $\mathcal{K}_H$ normalized so that $\mathrm{meas}(\mathcal{K}_H) = 1$. (The equalities on the second and fourth lines come from routine Haar measure manipulations. The interchange of integrals on the third line is justified by Lemma 6.3.) Thus,

$$(6.6)\quad \Theta_\pi(\gamma) = \phi_d(\gamma)\sum_{g\in K_\sigma\backslash G/H}\frac{\deg(\pi)}{\deg(\sigma)}\int_{K_\sigma gH/Z(G)}\int_{\mathcal{K}_H}\dot\theta_\sigma(^{g'k}\gamma)dk\,d\dot g'.$$

Fix a double coset $K_\sigma gH$ in G. Since $g'\mapsto\dot\theta_\sigma(^{g'k}\gamma)$ is invariant under left translation by K_σ, we have that

$$(6.7)\quad \int_{K_\sigma gH/Z(G)}\int_{\mathcal{K}_H}\dot\theta_\sigma(^{g'k}\gamma)dk\,d\dot g'$$

$$= \int_{K_\sigma(^gH)/Z(G)}\int_{^g\mathcal{K}_H}\dot\theta_\sigma(^{yk'g}\gamma)dk'\,d\dot y$$

$$= \int_{K_\sigma(^gH)/^gH}\int_{^gH/Z(G)}\int_{^g\mathcal{K}_H}\dot\theta_\sigma(^{yh'k'g}\gamma)dk'\,d\dot h'\,\frac{dy}{dh}$$

$$= \mathrm{meas}(K_\sigma(^gH)/^gH)\int_{^gH/Z(G)}\int_{^g\mathcal{K}_H}\dot\theta_\sigma(^{h'k'g}\gamma)dk\,d\dot h',$$

where $d\dot y$ is the Haar measure on $G/Z(G)$ used to compute $\deg(\pi)$, $d\dot h'$ is Haar measure on $^gH/Z(G)$ normalized as in the statement of the theorem, dk' is the Haar measure on $^g\mathcal{K}_H$ normalized so that $\mathrm{meas}(^g\mathcal{K}_H) = 1$, and dy/dh is the Haar measure on G/H deduced from $d\dot y$ and $d\dot h$. If there is no element g' in the double coset such that $^{g'}\gamma_{<r}\in G'$ and $x\in\mathcal{B}_r(^{g'}\gamma)$, then, by Corollary 4.5, the summand in (6.6) corresponding to g vanishes. Otherwise, we may, and do, assume that $^g\gamma_{<r}\in G'$ and $x\in\mathcal{B}_r(^g\gamma)$. Now, by Corollary 4.6,

$$\dot\theta_\sigma(^{h'k'g}\gamma) = \theta_\sigma(^g\gamma_{<r})\cdot[(^gH)_{x,r}](^{h'k'g}\gamma_{\geq r})\hat\phi(^{h'k'g}\gamma_{\geq r})$$

for $h' \in {}^g H$ and $k' \in {}^g \mathcal{K}_H$; and

$$\operatorname{meas}(K_\sigma({}^g H)/{}^g H) = \operatorname{meas}(K_\sigma/Z(G)) \operatorname{meas}(K_\sigma \cap {}^g H/Z(G))^{-1} = \frac{\deg(\sigma)}{\deg(\pi)},$$

where the last equality follows from the normalization $\operatorname{meas}(K_\sigma \cap {}^g H/Z(G)) = 1$ and the fact that $\pi = \operatorname{Ind}_{K_\sigma}^G \sigma$. Combining these two facts with Lemma B.4 (with $Z = Z(G)$) and (6.7), we see that the summands in (6.2) and (6.6) corresponding to g are the same. $\qquad\square$

We would like a way of describing the sum in Theorem 6.4 as running over a set of conjugates of γ, not over a set of elements conjugating γ. However, really we are interested only in conjugates of $\gamma_{<r}$, not of γ. We define below an equivalence relation $\overset{d-1}{\sim}$ on the set $\mathcal{T}((\mathbf{G}^i, \ldots, \mathbf{G}^d), (r_i, \ldots, r_d))$ that makes this precise, and then sum over equivalence classes for this relation in Corollary 6.6. Since we will need them later, in fact we define a family of equivalence relations $\overset{i}{\sim}$.

Definition 6.5. For $0 \le i < d$, let $\overset{i}{\sim}$ be the equivalence relation on $\mathcal{T}((\mathbf{G}^i, \ldots, \mathbf{G}^d), (r_i, \ldots, r_d))$ such that, for two elements δ and δ' of that set, $\delta \overset{i}{\sim} \delta'$ if and only if $\delta'_{<r_j} \in {}^{\operatorname{stab}_{G^j}(\overline{x})} \delta_{<r_j}$ for all $i \le j < d$.

Corollary 6.6. *If $x \in \mathcal{B}_r(\gamma)$, then*

$$(6.8) \qquad \theta_\tau(\gamma) = \phi_d(\gamma) \sum \theta_\sigma(\gamma'_{<r}) \hat{\mu}_{X^*}^{\operatorname{stab}_{H'}(\overline{x})}(\mathsf{e}_x^{-1}(\gamma'_{\ge r})).$$

If $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic and γ is regular semisimple, then

$$(6.9) \qquad \Theta_\pi(\gamma) = \phi_d(\gamma) \sum \theta_\sigma(\gamma'_{<r}) \hat{\mu}_{X^*}^{H'}(\mathsf{e}_x^{-1}(\gamma'_{\ge r})).$$

Here, $\hat{\mu}_{X^*}^{\operatorname{stab}_{H'}(\overline{x})}$ and $\hat{\mu}_{X^*}^{H'}$ are defined with respect to the Haar measure on $H'/Z(G)$ normalized so that $\operatorname{meas}(K_\sigma \cap H'/Z(G)) = 1$, and the sums are taken over $\overset{d-1}{\sim}$-equivalence classes of elements $\gamma' \in {}^{\operatorname{stab}_G(\overline{x})}\gamma \cap \mathcal{T}((\mathbf{G}', \mathbf{G}), (r, r_d))$ (respectively, $\gamma' \in {}^G\gamma \cap \mathcal{T}((\mathbf{G}', \mathbf{G}), (r, r_d))$ with $x \in \mathcal{B}_r(\gamma')$).

The notations $\mathcal{T}((\mathbf{G}', \mathbf{G}), (r, r_d))$ and $\overset{d-1}{\sim}$ are as in Definitions 1.4.1 and 6.5, respectively. By abuse of notation, we have written $\mathbf{H}'$ in place of $C_{\mathbf{G}}(\gamma'_{<r_{d-1}})$, even though this group depends on γ'.

As observed after Theorem 6.4, by Lemma B.2, we may describe the orbital integrals over the possibly disconnected groups H' as sums of orbital integrals over H'°.

Proof. Let $\mathcal{G}$ be a subgroup of G containing $\operatorname{stab}_{G'}(\overline{x})G_{x,0+}$, and put $\mathcal{H} = H \cap \mathcal{G}$ and $\mathcal{C} = \{ g \in \mathcal{G} \mid {}^g\gamma \in \mathcal{T}((\mathbf{G}', \mathbf{G}), (r, r_d)) \text{ and } x \in \mathcal{B}_r({}^g\gamma) \}$.

First, we claim that the natural map $f_1 : \operatorname{stab}_{G'}(\overline{x}) \backslash \mathcal{G}/\mathcal{H} \longrightarrow \operatorname{stab}_{G'}(\overline{x}) G_{x,0+} \backslash \mathcal{G}/\mathcal{H}$ furnishes a bijection of $\operatorname{stab}_{G'}(\overline{x}) \backslash \mathcal{C}/\mathcal{H}$ with the set of $(\operatorname{stab}_{G'}(\overline{x}) G_{x,0+}, \mathcal{H})$-double cosets containing an element of $\mathcal{C}$. The map is clearly surjective, so it suffices to show that it is injective. Suppose that $g_1, g_2 \in \mathcal{C}$ are such that

$$\operatorname{stab}_{G'}(\overline{x}) G_{x,0+} g_1 \mathcal{H} = \operatorname{stab}_{G'}(\overline{x}) G_{x,0+} g_2 \mathcal{H}.$$

Since $\operatorname{stab}_{G'}(\overline{x}) G_{x,0+} = G_{x,0+} \operatorname{stab}_{G'}(\overline{x})$, we have that

$$G_{x,0+} g_1 \mathcal{H} \cap \operatorname{stab}_{G'}(\overline{x}) g_2 \mathcal{H} \neq \emptyset$$

— say $k g_1 \mathcal{H} = g' g_2 \mathcal{H}$, with $k \in G_{x,0+}$ and $g' \in \operatorname{stab}_{G'}(\overline{x})$. Then ${}^{kg_1}\gamma_{<r} = {}^{g'g_2}\gamma_{<r} \in G'$. Since ${}^{g_1}\gamma_{<r} \in G'$ and $x \in \mathcal{B}_r({}^{g_1}\gamma_{<r})$, we have by Lemma 9.10 and Corollary 6.14 of [6] that $k \in G'_{x,0+}({}^{g_1}H)_{x,0+}$. In particular, $k g_1 \in \operatorname{stab}_{G'}(\overline{x}) g_1 H$. Since also $k g_1 \in \mathcal{G}$, we have

$$k g_1 \in \operatorname{stab}_{G'}(\overline{x}) g_1 H \cap \mathcal{G} \subseteq \operatorname{stab}_{G'}(\overline{x}) g_1 (H \cap \mathcal{G}) = \operatorname{stab}_{G'}(\overline{x}) g_1 \mathcal{H}.$$

Since $g' g_2 \in k g_1 \mathcal{H}$, we have that g_2 belongs to the same $(\operatorname{stab}_{G'}(\overline{x}), \mathcal{H})$-double coset as g_1, as desired.

Second, notice that the map f_2 on $\operatorname{stab}_{G'}(\overline{x}) \backslash \mathcal{C}/H$ that sends a double coset $\operatorname{stab}_{G'}(\overline{x}) g \mathcal{H}$ to the $\operatorname{stab}_{G'}(\overline{x})$-orbit of ${}^g\gamma_{<r}$ is a well defined injection.

Third, consider the map f_3 from $\mathcal{S} = \left\{ \gamma' \in {}^{\mathcal{G}}\gamma \cap \mathcal{T}((\mathbf{G}', \mathbf{G}), (r, r_d)) \;\middle|\; x \in \mathcal{B}_r(\gamma') \right\}$ to the set of $\operatorname{stab}_{G'}(\overline{x})$-orbits in ${}^{\mathcal{G}}\gamma_{<r} \cap G'$ that sends an element $\gamma' \in \mathcal{S}$ to the $\operatorname{stab}_{G'}(\overline{x})$-orbit of $\gamma'_{<r}$. By definition, two elements $\gamma', \gamma'' \in {}^{\mathcal{G}}\gamma \cap \mathcal{T}((\mathbf{G}', \mathbf{G}), (r, r_d))$ have the same image if and only if $\gamma' \overset{d-1}{\sim} \gamma''$. Thus, the induced map on $\overset{d-1}{\sim}$-equivalence classes in $\mathcal{S}$ is an injection. It is easy to see that the images of f_2 and f_3 are the same.

Now we consider the composition $f_3^{-1} \circ f_2 \circ f_1^{-1}$. This furnishes a bijection of the set of $(\operatorname{stab}_{G'}(\overline{x}) G_{x,0+}, \mathcal{H})$-double cosets containing an element of $\mathcal{C}$ into the set of $\overset{d-1}{\sim}$-equivalence classes in $\mathcal{S}$. If $\mathcal{G} = \operatorname{stab}_G(\overline{x})$, then $\mathcal{H} = \operatorname{stab}_H(\overline{x})$; the specified set of double cosets is the indexing set for the sum in (6.1); and the set of equivalence classes is the indexing set for the sum in (6.8). It is easy to check that the summands match term-by-term, so (6.8) holds. Similarly, we demonstrate (6.9) by taking $\mathcal{G} = G$ (and observing that $\operatorname{stab}_{G'}(\overline{x}) = G'$ when $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic). $\qquad\square$

We now prove a single-orbit result in the spirit of Murnaghan–Kirillov theory (see [4, 19, 47–50]). When F has characteristic zero and p is large, the second statement is a special case of Theorem 5.3.1 of [37].

Corollary 6.7. *Suppose that there exists a bijection* $\mathsf{e} : \bigcup_{y \in \mathcal{B}(\mathbf{G}, F)} \mathfrak{g}_{y,0+} \longrightarrow \bigcup_{y \in \mathcal{B}(\mathbf{G}, F)} G_{y,0+}$ *such that, for all* $y \in \mathcal{B}(\mathbf{G}, F)$, *the restriction* $\mathsf{e}\big|_{\mathfrak{g}_{y,0+}}$ *has image in* $G_{y,0+}$ *and satisfies Hypothesis A.7 (for all tame maximal F-tori* $\mathbf{T}$ *with* $y \in \mathcal{B}(\mathbf{T}, F)$).

Fix $\gamma \in G$ such that $\gamma \in G_{y,r}$ for some $y \in \mathcal{B}(\mathbf{G}, F)$. If $\gamma \in G_{x,r}$ (i.e., if we may take $y = x$), then

$$\theta_\tau(\gamma) = \phi_d(\gamma)[\mathrm{stab}_G(\overline{x}) : K_\sigma]^{-1} \deg(\tau)\hat{\mu}_{X^*}^{\mathrm{stab}_G(\overline{x})}(\mathsf{e}^{-1}(\gamma)).$$

If $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic and γ is regular semisimple, then

$$\Theta_\pi(\gamma) = \phi_d(\gamma) \deg(\pi)\hat{\mu}_{X^*}^{G}(\mathsf{e}^{-1}(\gamma)).$$

Here, $\hat{\mu}_{X^*}^{\mathrm{stab}_G(\overline{x})}$ and $\hat{\mu}_{X^*}^{G}$ are defined with respect to the Haar measure on $G/Z(\mathbf{G})$ normalized so that $\mathrm{meas}(K_\sigma/Z(G)) = 1$.

Proof. Note that γ trivially has a normal r-approximation, and that $\gamma_{<r} = 1$, so $\mathbf{H} = \mathbf{G}$. Since $\gamma = \gamma_{\geq r}$, we have that $x \in \mathcal{B}_r(\gamma)$ if and only if $\gamma \in G_{x,r}$.

Put $Y = \mathsf{e}^{-1}(\gamma)$. If ${}^{G}\gamma \cap G_{x,r} = \emptyset$, then, by Hypothesis A.7(2), ${}^{G}Y \cap \mathfrak{g}_{x,r} = \emptyset$. Therefore, by Lemma B.4, $\hat{\mu}_{X^*}^{G}(Y) = 0$. If also $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic, then, by Theorem 6.4, we have that $\dot\theta_\sigma({}^{g}\gamma) = 0$ for $g \in G$, hence (by the Frobenius formula) that $\Theta_\pi(\gamma) = 0$.

Thus we may, and do, assume that $\gamma \in G_{x,r}$. In particular, equation (6.1) holds, and the sum on the right-hand side of that equation has a single summand, so it becomes

$$(6.1')\qquad \theta_\tau(\gamma) = \phi_d(\gamma)\theta_\sigma(1)\hat{\mu}_{X^*}^{\mathrm{stab}_G(\overline{x})}(Y) = \phi_d(\gamma) \deg(\sigma)\hat{\mu}_{X^*}^{\mathrm{stab}_G(\overline{x})}(Y).$$

Since $\tau = \mathrm{Ind}_{K_\sigma}^{\mathrm{stab}_G(\overline{x})} \sigma \otimes \phi_d$, we have that $\deg(\tau) = [\mathrm{stab}_G(\overline{x}) : K_\sigma] \deg(\sigma)$, so

$$\theta_\tau(\gamma) = \phi_d(\gamma)[\mathrm{stab}_G(\overline{x}) : K_\sigma]^{-1} \deg(\tau)\hat{\mu}_{X^*}^{\mathrm{stab}_G(\overline{x})}(Y).$$

The second equality follows similarly from (6.2) and the fact that $\deg(\pi) = \mathrm{meas}(K_\sigma/Z(G)) \deg(\sigma) = \deg(\sigma)$. $\qquad\square$

7. THE FULL CHARACTER FORMULA

Here we unroll the inductive formulas from §§5 and 6, preserving the hypotheses of §6. In particular, γ is an element of G with a normal r_{d-1}-approximation, which we fixed for definiteness. Thus, the elements $\gamma_{<r_i}$ are unambiguously defined for $0 \leq i < d$. Choosing such an approximation also fixes approximations to all truncations and conjugates of γ.

Theorem 7.1. *If $x \in \mathcal{B}_r(\gamma)$, then*
(7.1)

$$\theta_\tau(\gamma) = \phi_d(\gamma) \sum c(\vec{\phi}, \gamma'_{<r_{d-1}}) \left(\prod_{i=0}^{d-1} \mathfrak{G}(\phi_i, \gamma'_{<r_i})\varepsilon(\phi_i, \gamma'_{<r_i}) \right)$$

$$\times \left(\prod_{i=0}^{d-1} \phi_i(\gamma'_{<r_i}) \right) \theta_{\rho'_0}(\gamma'_0) \prod_{i=0}^{d-1} \hat{\mu}_{X_i^*}^{\mathrm{stab}_{H^{i\prime}}(\overline{x})}(\mathsf{e}_x^{-1}(\gamma'_{(i)})).$$

If $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic and $\gamma \in G$ is regular semisimple, then

$$\Theta_\pi(\gamma) = \phi_d(\gamma) \sum c(\vec{\phi}, \gamma'_{<r_{d-1}}) \Big(\prod_{i=0}^{d-1} \mathfrak{G}(\phi_i, \gamma'_{<r_i}) \varepsilon(\phi_i, \gamma'_{<r_i}) \Big)$$

(7.2)

$$\times \Big(\prod_{i=0}^{d-1} \phi_i(\gamma'_{<r_i}) \Big) \Theta_{\pi'_0}(\gamma'_0) \prod_{i=0}^{d-1} \hat{\mu}_{X_i^*}^{H^{i\prime}}(\mathsf{e}_x^{-1}(\gamma'_{(i)})).$$

Here,

$$c(\vec{\phi}, \gamma'_{<r_{d-1}}) = \prod_{i=0}^{d-1} \Big[[\![\gamma'_{<r_i}; x, r_i]\!]_{G^{i+1}} : [\![\gamma'_{<r_i}; x, r_i]\!]_{G^i} G_{x,s_i}^{i+1} \Big]^{1/2}$$

$$\times \Big[[\![\gamma'_{<r_i}; x, r_i +]\!]_{G^{i+1}} : [\![\gamma'_{<r_i}; x, r_i +]\!]_{G^i} G_{x,s_i+}^{i+1} \Big]^{1/2},$$

and

- $\hat{\mu}_{X_i^*}^{\mathrm{stab}_{H^{i\prime}}(\overline{x})}$ *and* $\hat{\mu}_{X_i^*}^{H^{i\prime}}$ *are defined with respect to the Haar measure on* $H^{i\prime}/Z(G)$ *normalized so that* $\mathrm{meas}(K_{\sigma_{i+1}} \cap H^{i\prime}/Z(G)) = 1$,
- $\gamma'_{(i)} = (\gamma'_{<r_{i+1}})_{\geq r_i}$ *when* $0 \leq i < d-1$, *and*
- $\gamma'_{(d-1)} = \gamma'_{\geq r_{d-1}}$.

The sums are taken over $\overset{0}{\sim}$-*equivalence classes of elements* $\gamma' \in {}^{\mathrm{stab}_G(\overline{x})}\gamma \cap \mathcal{T}(\vec{\mathbf{G}}, \vec{r})$ *(respectively,* $\gamma' \in {}^G\gamma \cap \mathcal{T}(\vec{\mathbf{G}}, \vec{r})$ *such that* $x \in \mathcal{B}_{r_{d-1}}(\gamma')$).

The notations $\mathcal{T}(\vec{\mathbf{G}}, \vec{r})$ and $\overset{0}{\sim}$ are as in Definitions 1.4.1 and 6.5, respectively. As in Corollary 6.6, we have written $\mathbf{H}^{i\prime}$ in place of $C_{\mathbf{G}^{i+1}}(\gamma'_{<r_i})$ for $0 \leq i < d$.

Note that, in particular, if $\mathbf{G}'/Z(\mathbf{G})$ is F-anisotropic, then the character of π is supported on conjugacy classes intersecting $\mathcal{T}(\vec{\mathbf{G}}, \vec{r})$. A similar statement holds for the character of τ.

Recall that the various roots of unity $\mathfrak{G}$ were defined and computed in §5.2.

As observed after Theorem 6.4, by Lemma B.2, we may describe the orbital integrals over possibly disconnected groups in the above formula as sums of orbital integrals over connected groups.

Proof. In this proof only, we write r_d for ∞. This conflicts with the notation in the rest of the paper, but it makes the equations appearing below (for example, (7.3)) simpler.

For $0 \leq i < d$, we may apply Proposition 5.3.3 and Corollary 6.6, with $(\mathbf{G}^i, \mathbf{G}^{i+1})$ in place of $(\mathbf{G}', \mathbf{G})$, to see that, for all $\delta \in \mathrm{stab}_{G^{i+1}}(\overline{x})$ such that

$x \in \mathcal{B}_{r_i}(\delta)$,

$(\ast_i)$

$$\theta_{\tau_{i+1}}(\delta) = \phi_{i+1}(\delta) \sum \Big[[\![\delta_{<r_i}; x, r_i]\!]_{G^{i+1}} : [\![\delta_{<r_i}; x, r_i]\!]_{G^i} G^{i+1}_{x,s_i} \Big]^{1/2}$$

$$\times \Big[[\![\delta_{<r_i}; x, r_i+]\!]_{G^{i+1}} : [\![\delta_{<r_i}; x, r_i+]\!]_{G^i} G^{i+1}_{x,s_i+} \Big]^{1/2}$$

$$\times \mathfrak{G}(\phi_i, \delta'_{<r_i}) \varepsilon(\phi_i, \delta'_{<r_i})$$

$$\times \theta_{\tau_i}(\delta'_{<r_i}) \hat{\mu}^{\mathrm{stab}_{H^{i\prime}}(\overline{x})}_{X_i^*}(\mathsf{e}_x^{-1}(\delta'_{\geq r_i})),$$

the sum taken over $\overset{i}{\sim}$-equivalence classes of elements $\delta' \in {}^{\mathrm{stab}_{G^{i+1}}(\overline{x})}\delta \cap \mathcal{T}((\mathbf{G}^i, \mathbf{G}^{i+1}), (r_i, r_{i+1}))$. (The condition $x \in \mathcal{B}_{r_i}(\delta')$ is automatically satisfied here.) Note that the set $\mathcal{B}_{r_i}(\delta)$ and the equivalence relation $\overset{i}{\sim}$ are both constructed in the setting of some ambient group, which is suppressed from the notation. However, it is easy to see that changing the ambient group from $\mathbf{G}$ to $\mathbf{G}^{i+1}$ corresponds simply to restricting the equivalence relation $\overset{i}{\sim}$; and, since $x \in \mathcal{B}(\mathbf{G}^{i+1}, F)$, Lemma 8.2 of [6] shows that we do not need to worry about what is the ambient group for the construction of $\mathcal{B}_{r_i}(\delta)$.

Put $\gamma^{(0)} = \gamma$. We apply $(\ast_{d-1})$ to describe $\theta_\tau(\gamma) = \theta_{\tau_d}(\gamma)$ in terms of the values of $\theta_{\tau_{d-1}}$ at truncations of various conjugates $\gamma^{(1)}$ of $\gamma^{(0)} = \gamma^{(0)}_{<r_d}$; then $(\ast_{d-2})$ to describe each $\theta_{\tau_{d-1}}(\gamma^{(1)})$ in terms of the values of $\theta_{\tau_{d-2}}$ at truncations of various conjugates $\gamma^{(2)}$ of $\gamma^{(1)}_{<r_{d-1}}$; and so forth to obtain

(7.3)

$$\theta_\tau(\gamma) = \phi_d(\gamma) \sum \underbrace{\Big(\prod_{i=0}^{d-1} \Big[[\![\gamma^{(d-i)}_{<r_i}; x, r_i]\!]_{G^{i+1}} : [\![\gamma^{(d-i)}_{<r_i}; x, r_i]\!]_{G^i} G^{i+1}_{x,s_i} \Big]^{1/2} \Big)}_{\mathrm{I}}$$

$$\times \underbrace{\Big(\prod_{i=0}^{d-1} \Big[[\![\gamma^{(d-i)}_{<r_i}; x, r_i+]\!]_{G^{i+1}} : [\![\gamma^{(d-i)}_{<r_i}; x, r_i+]\!]_{G^i} G^{i+1}_{x,s_i+} \Big]^{1/2} \Big)}_{\mathrm{I}'}$$

$$\times \underbrace{\Big(\prod_{i=0}^{d-1} \mathfrak{G}(\phi_i, \gamma^{(d-i)}_{<r_i}) \varepsilon(\phi_i, \gamma^{(d-i)}_{<r_i}) \Big)}_{\mathrm{II}}$$

$$\times \underbrace{\Big(\prod_{i=0}^{d-2} \phi_{i+1}(\gamma^{(d-i)}) \Big) \theta_{\tau_0}(\gamma^{(d)}_{<r_0}) \prod_{i=0}^{d-1} \hat{\mu}^{\mathrm{stab}_{H^{i\prime}}(\overline{x})}_{X_i^*}(\mathsf{e}_x^{-1}(\gamma^{(d-i)}_{\geq r_i}))}_{\mathrm{IV}},$$

the sum taken over the collection $\mathcal{S}$ of d-tuples $\big([\gamma^{(d-i)}]_i\big)_{i=0}^{d-1}$ with $\gamma^{(d-i)} \in$ $^{\mathrm{stab}_{G^{i+1}}(\overline{x})}\gamma_{<r_{i+1}}^{(d-i-1)} \cap \mathcal{T}((\mathbf{G}^i, \mathbf{G}^{i+1}), (r_i, r_{i+1}))$ for $0 \le i < d$. (Here, $[\delta]_i$ denotes the $\overset{i}{\sim}$-equivalence class of an element δ for $0 \le i < d$.)

If $\gamma' \in {}^{\mathrm{stab}_G(\overline{x})}\gamma \cap \mathcal{T}(\vec{\mathbf{G}}, \vec{r})$, then $S(\gamma') := \big([\gamma'_{<r_{i+1}}]_i\big)_{i=0}^{d-1}$ lies in $\mathcal{S}$. It is an easy consequence of the definitions that, for $\gamma'' \in {}^{\mathrm{stab}_G(\overline{x})}\gamma \cap \mathcal{T}(\vec{\mathbf{G}}, \vec{r})$, we have $S(\gamma') = S(\gamma'')$ if and only if $\gamma' \overset{0}{\sim} \gamma''$. On the other hand, suppose that $\vec{\gamma}' = \big([\gamma^{(d-i)}]_i\big)_{i=0}^{d-1} \in \mathcal{S}$. Then, by definition, there are elements $g_i \in \mathrm{stab}_{G^{i+1}}(\overline{x})$ such that $\gamma^{(d-i)} = {}^{g_i}\gamma_{<r_{i+1}}^{(d-i-1)}$ for $0 \le i \le d$. One checks that $\gamma' := {}^{g_0\cdots g_{d-1}}\gamma \in {}^{\mathrm{stab}_G(\overline{x})}\gamma \cap \mathcal{T}(\vec{\mathbf{G}}, \vec{r})$, and $S(\gamma') = \vec{\gamma}'$. Thus, S induces a bijection from the set of $\overset{0}{\sim}$-equivalence classes in $^{\mathrm{stab}_G(\overline{x})}\gamma \cap \mathcal{T}(\vec{\mathbf{G}}, \vec{r})$ onto $\mathcal{S}$, so that we may regard the sum in (7.3) as running over the former set. Upon doing so, we notice that the product of terms (I) and (I$'$) becomes $c(\vec{\phi}, \gamma'_{<r_{d-1}})$. We calculate the remaining terms appearing in (7.3) as follows.

(II) This matches with the corresponding term in (7.1).

(III) Since $\theta_{\tau_0}(\gamma'_{<r_0}) = \phi_0(\gamma'_{<r_0})\theta_{\rho'_0}(\gamma'_{<r_0})$ and $\theta_{\rho'_0}(\gamma'_{<r_0}) = \theta_{\rho'_0}(\gamma'_0)$, this becomes $\big(\prod_{i=0}^{d-1}\phi_i(\gamma'_{<r_i})\big)\theta_{\rho'_0}(\gamma'_0)$.

(IV) When we replace $\gamma^{(d-i)}$ by $\gamma'_{<r_{i+1}}$, the element $\gamma^{(d-i)}_{\ge r_i}$ becomes $(\gamma'_{<r_{i+1}})_{\ge r_i} = \gamma'_{(i)}$, even when $i = d-1$ (because we have set $r_d = \infty$ in this proof). Thus, this matches up with the corresponding term in (7.1).

Since (7.3) holds, and can be matched term-by-term with (7.1), we also have that (7.1) holds.

The argument carries over essentially unchanged to prove (7.2) holds. We sketch the few minor differences. Instead of using $(*_{d-1})$, we apply Proposition 5.3.3 and Corollary 6.6 to obtain

$(**_{d-1})$
$$\Theta_\pi(\gamma) = \phi_d(\gamma) \sum \Big[[\![\gamma^{(1)}_{<r_{d-1}}; x, r_{d-1}]\!] : [\![\gamma^{(1)}_{<r_{d-1}}; x, r_{d-1}]\!]_{G^{d-1}} G^d_{x, s_{d-1}} \Big]^{1/2}$$
$$\times \Big[[\![\gamma^{(1)}_{<r_{d-1}}; x, r_{d-1}+]\!] : [\![\gamma^{(1)}_{<r_{d-1}}; x, r_{d-1}+]\!]_{G^{d-1}} G^d_{x, s_{d-1}+} \Big]^{1/2}$$
$$\times \mathfrak{G}(\phi_{d-1}, \gamma^{(1)}_{<r_{d-1}})\varepsilon(\phi, \gamma^{(1)}_{<r_{d-1}})$$
$$\times \theta_{\tau_{d-1}}(\gamma^{(1)}_{<r_{d-1}})\hat{\mu}_{X^*_{d-1}}^{H^{d-1}{}'}(\mathsf{e}_x^{-1}(\gamma^{(1)}_{\ge r_{d-1}})),$$

the sum taken over $\overset{d-1}{\sim}$-equivalence classes of elements $\gamma^{(1)} \in {}^G\gamma \cap \mathcal{T}((\mathbf{G}', \mathbf{G}), (r, r_d))$ such that $x \in \mathcal{B}_{r_{d-1}}(\gamma^{(1)})$. We then apply $(*_{d-2})$, ..., $(*_0)$ as before to obtain a sum over a collection of d-tuples $\big([\gamma^{(d-i)}]_i\big)_{i=0}^{d-1}$, but this time we require only that $\gamma^{(d-i)} \in {}^{G^{i+1}}\gamma_{<r_{i+1}}^{(d-i-1)} \cap \mathcal{T}((\mathbf{G}^i, \mathbf{G}^{i+1}), (r_i, r_{i+1}))$ and $x \in$

$\mathcal{B}_{r_i}(\gamma^{(d-i)})$ for $0 \leq i < d$ (that is, we allow $\gamma^{(d-i)}$ to range over a G^{i+1}-, not just a $\mathrm{stab}_{G^{i+1}}(\overline{x})$-, orbit). The key point here is that $G^{i+1} = \mathrm{stab}_{G^{i+1}}(\overline{x})$ unless $i = d-1$. Again, we use (a natural extension of) the map S to identify this collection of d-tuples with the set of $\overset{0}{\sim}$-equivalence classes of elements $\gamma' \in {}^G\gamma \cap \mathcal{T}(\vec{\mathbf{G}}, \vec{r})$ with $x \in \mathcal{B}_r(\gamma')$. Finally, we note that $\rho'_0 = \pi'_0$. $\square$

APPENDIX A. MOCK-EXPONENTIAL MAP

In this section, $\mathbf{G}$ is a reductive algebraic F-group (possibly disconnected) that splits over a tame extension of F. We state below two useful hypotheses (Hypotheses A.1 and A.7) regarding mock-exponential maps. Proposition A.8 will show that the first always holds, and the second holds when $\mathbf{G}$ is connected. (Note that, with appropriate modifications in the statement of Hypothesis A.1(6), these hypotheses also make sense for groups that are not tame. However, they are not always satisfied for such groups.)

Hypothesis A.1. There exists a family

$$(\mathsf{e}^E_{x,t:u} : \mathrm{Lie}(\mathbf{G})(E)_{x,t:u} \longrightarrow \mathbf{G}(E)_{x,t:u})_{\substack{E/F \text{ finite and tamely ramified} \\ x \in \mathcal{B}(\mathbf{G},E) \\ t,u \in \widetilde{\mathbb{R}}_{>0} \\ t \leq u \leq 2t}}$$

of isomorphisms such that, given

- a finite, tamely ramified extension E/F,
- $x \in \mathcal{B}(\mathbf{G}, E)$,
- $t_1, t_2, u_1 \in \widetilde{\mathbb{R}}_{>0}$ with $t_1 \leq u_1 \leq 2t_1$,
- $X_j \in \mathrm{Lie}(\mathbf{G})(E)_{x,t_j}$ for $j = 1, 2$, and
- $g_j \in \mathsf{e}^E_{x,t_j:t_j+}(X_j)$ for $j = 1, 2$,

the following statements hold.

(1) If L is a field intermediate between E and F and $x \in \mathcal{B}(\mathbf{G}, L)$, then the restriction of $\mathsf{e}^E_{x,t_1:u_1}$ to $\mathrm{Lie}(\mathbf{G})(L)_{x,t_1:u_1}$ is $\mathsf{e}^L_{x,t_1:u_1}$.

(2) If E/F is Galois, then $\mathsf{e}^E_{x,t_1:u_1}$ is $\mathrm{Gal}(E/F)$-equivariant.

(3) If $u_2 \in \widetilde{\mathbb{R}}_{>0}$ and $t_1 \leq t_2 \leq u_2 \leq u_1 \leq 2t_1$, then $\mathsf{e}^E_{x,t_1:u_1}(X_2) \subseteq \mathsf{e}^E_{x,t_2:u_2}(X_2)$.

(4) $(\mathrm{Ad}(g_1) - 1)X_2 \in (\mathsf{e}^E_{x,(t_1+t_2):(t_1+t_2)+})^{-1}[g_1, g_2]$.

(5) $[X_1, X_2] \in (\mathsf{e}^E_{x,(t_1+t_2):(t_1+t_2)+})^{-1}[g_1, g_2]$.

(6) If
- $\mathbf{T}$ is a tame maximal E-torus,
- $x \in \mathcal{B}(\mathbf{T}, E)$,
- f is a $\mathrm{Gal}(F^{\mathrm{sep}}/E)$-invariant concave function on $\widetilde{\Phi}(\mathbf{G}, \mathbf{T})$, and
- $t_1 \leq f(\alpha) \leq u_1$ for all $\alpha \in \widetilde{\Phi}(\mathbf{G}, \mathbf{T})$,

then $\mathsf{e}^E_{x,t_1:u_1}$ restricts to an isomorphism of the image in $\mathrm{Lie}(\mathbf{G})(E)_{x,t_1:u_1}$ of $_{\mathbf{T}}\,\mathrm{Lie}(\mathbf{G})(E)_{x,f}$ with the image in $\mathbf{G}(E)_{x,t_1:u_1}$ of $_{\mathbf{T}}\mathbf{G}(E)_{x,f}$.

(For convenience, we have regarded $\mathsf{e}^E_{x,t_1:u_1}$ also as a function on $\mathrm{Lie}(\mathbf{G})(E)_{x,t_1}$.)

We will often suppress a superscript F, writing $\mathsf{e}_{x,t:u}$ instead of $\mathsf{e}^F_{x,t:u}$.

Remark A.2. Note that, with the notation of Hypothesis A.1(4), $\mathrm{Ad}(h)X \equiv X \pmod{\mathrm{Lie}(\mathbf{G})(E)_{x,t_1+t_2}}$. Analogous statements for Int and ad are proven in Proposition 1.4.1 of [1].

Remark A.3. Given Hypothesis A.1(1), it suffices to verify Hypotheses A.1(2)–A.1(6) only for E "sufficiently large tame Galois", in the sense that, given a finite, tamely ramified field extension E/F, there is a finite, tamely ramified Galois superextension M/F of E/F for which the hypotheses hold. Indeed, it is a straightforward application of Hypothesis A.1(1) that, if Hypotheses A.1(2)–A.1(5) hold for such an M, then they hold for E as well.

Now let E, x, t_1, and u_1 be as usual, and let $\mathbf{T}$ and f be as in Hypothesis A.1(6). Let M/F be a (finite, tame) Galois superextension of E/F such that Hypothesis A.1(6) is satisfied for M, x, t_1, u_1, $\mathbf{T}$, and f. By further enlarging M if necessary, we may, and do, assume that $\mathbf{G}$ is M-split.

Let $\overline{u}_1$ be the constant function on $\widetilde{\Phi}(\mathbf{G}, \mathbf{T})$ with value u_1. By assumption, $\mathsf{e}^M_{x,t_1:u_1}$ induces an isomorphism of $_{\mathbf{T}}\,\mathrm{Lie}(\mathbf{G})(M)_{x,f:\overline{u}_1}$ with $_{\mathbf{T}}\mathbf{G}(M)_{x,f:\overline{u}_1}$. By Hypothesis A.1(2), the isomorphism is $\mathrm{Gal}(M/F)$-, hence certainly $\mathrm{Gal}(M/E)$-, equivariant; so we obtain an isomorphism $_{\mathbf{T}}\,\mathrm{Lie}(\mathbf{G})(M)^{\mathrm{Gal}(M/E)}_{x,f:\overline{u}_1} \longrightarrow {}_{\mathbf{T}}\mathbf{G}(M)^{\mathrm{Gal}(M/E)}_{x,f:\overline{u}_1}$. By Proposition 5.39 of [6], we have $H^1(M/F, {}_{\mathbf{T}}\mathbf{G}(M)_{x,\overline{u}_1}) = \{0\}$, so that the codomain of the isomorphism is $_{\mathbf{T}}\mathbf{G}(M)^{\mathrm{Gal}(M/E)}_{x,f}/{}_{\mathbf{T}}\mathbf{G}(M)^{\mathrm{Gal}(M/E)}_{x,\overline{u}_1}$. By Lemma 5.33 of *loc. cit.*, this is just $_{\mathbf{T}}\mathbf{G}(E)_{x,f:\overline{u}_1}$. A similar, but easier, calculation shows that the domain of the isomorphism is $_{\mathbf{T}}\,\mathrm{Lie}(\mathbf{G})(E)_{x,f:\overline{u}_1}$. Thus, Hypothesis A.1(6) is satisfied for E, x, t_1, u_1, $\mathbf{T}$, and f, as desired.

For the next three results, suppose that

- E/F is a finite, tamely ramified extension,
- $\mathbf{T}$ is a tame maximal E-torus in $\mathbf{G}$,
- $x \in \mathcal{B}(\mathbf{T}, E)$,
- $t, u \in \widetilde{\mathbb{R}}_{>0}$ with $t \le u \le 2t$, and
- $\mathsf{e}^E_{x,t:u}$ is as in Hypothesis A.1.

Lemma A.4. *Suppose that* $\mathbf{H}$ *is a reductive E-subgroup of* $\mathbf{G}$ *containing* $\mathbf{T}$. *Then* $\mathsf{e}^E_{x,t:u}$ *restricts to an isomorphism of* $\mathrm{Lie}(\mathbf{H})(E)_{x,t:u}$ *with* $\mathbf{H}(E)_{x,t:u}$.

Proof. Apply Hypothesis A.1(6), with f the concave function on $\widetilde{\Phi}(\mathbf{G}, \mathbf{T})$ that takes the value t on $\widetilde{\Phi}(\mathbf{H}, \mathbf{T})$ and u elsewhere. □

Lemma A.5. *If $Y \in \mathrm{Lie}(\mathbf{T})(E)_t$ and $h \in \mathsf{e}^E_{x,t:t+}(Y) \cap \mathbf{T}(E)$, then $\alpha(h) - 1 \equiv d\alpha(Y) \pmod{E_{t+}}$ for $\alpha \in \widetilde{\Phi}(\mathbf{G}, \mathbf{T})$.*

Proof. Fix $\alpha \in \widetilde{\Phi}(\mathbf{G}, \mathbf{T})$ and choose a non-zero, positive-depth element $X \in \mathrm{Lie}(\mathbf{G})(E)_\alpha$. Put $t' = \mathrm{d}_x(X)$, and let g be an element of $\mathsf{e}^E_{x,t':t'+}(X)$. Then, by Hypotheses A.1(4) and A.1(5), we have that

$$(\alpha(h) - 1)X = (\mathrm{Ad}(h) - 1)X$$

and

$$d\alpha(Y)X = [Y, X]$$

lie in $(\mathsf{e}^E_{x,(t+t'):(t+t')+})^{-1}[h, g]$, so $(\alpha(h)-1)X \equiv d\alpha(Y)X \pmod{\mathrm{Lie}(\mathbf{G})(E)_{x,(t+t')+}}$. Since $X \notin \mathrm{Lie}(\mathbf{G})(E)_{x,t'+}$, we have $\alpha(h) - 1 \equiv d\alpha(Y) \pmod{E_{t+}}$. $\square$

Lemma A.6. *Suppose that $\mathbf{T}$ is E-split and $\alpha \in \widetilde{\Phi}(\mathbf{G}, \mathbf{T})$. Then $\mathsf{e}^E_{x,t:u}$ induces an isomorphism of $_E\mathfrak{u}_{(\alpha+t):(\alpha+u)}$ with $_EU_{(\alpha+t):(\alpha+u)}$, where, for $c \in \widetilde{\mathbb{R}}$, $\alpha + c$ denotes the unique affine root with gradient α satisfying $(\alpha + c)(x) = c$.*

Proof. Apply Hypothesis A.1(6) with f the concave function on $\widetilde{\Phi}(\mathbf{G}, \mathbf{T})$ that takes the value t at α and u elsewhere. $\square$

Hypothesis A.7. There exists a family

$$(\mathsf{e}_{\mathbf{T},x} : \mathfrak{g}_{x,0+} \longrightarrow G_{x,0+})_{\substack{\mathbf{T} \text{ a tame maximal } F\text{-torus in } \mathbf{G} \\ x \in \mathcal{B}(\mathbf{T}, F)}}$$

of bijections such that, given

- a tame maximal F-torus $\mathbf{T}$ in $\mathbf{G}$,
- $x \in \mathcal{B}(\mathbf{T}, F)$,
- $t \in \widetilde{\mathbb{R}}_{>0}$, and
- $X \in \mathfrak{g}_{x,t}$,

the following statements hold.

(1) If $u \in \widetilde{\mathbb{R}}_{>0}$ with $t \le u \le 2t$, then $\mathsf{e}_{\mathbf{T},x}(X) \in \mathsf{e}_{x,t:u}(X)$.
(2) For $g \in G$, we have $^gX \in \mathfrak{g}_{x,t}$ if and only if $^g(\mathsf{e}_{\mathbf{T},x}(X)) \in G_{x,t}$, in which case $^g(\mathsf{e}_{\mathbf{T},x}(X)) \in \mathsf{e}_{x,t:t+}(^gX)$.
(3) If $\mathbf{H}$ is a reductive F-subgroup of $\mathbf{G}$ containing $\mathbf{T}$, then $\mathsf{e}_{\mathbf{T},x}(\mathfrak{h}_{x,0+}) = H_{x,0+}$.

Proposition A.8. *Hypothesis A.1 holds. If $\mathbf{G}$ is connected, then Hypothesis A.7 also holds.*

In fact, one can imitate the argument of Proposition 1.6.7 of [1] to show that Hypothesis A.7 holds even if $\mathbf{G}$ is not assumed to be connected; but we do not need to do so here.

Proof. The only difference between the hypotheses in question for $\mathbf{G}$ and $\mathbf{G}^\circ$ is in Hypothesis A.7(2), so it suffices to assume throughout that $\mathbf{G}$ is connected.

See [1, §1.5] for a description of a family of bijections

$$\left(\varphi_{\mathbf{T}(E),x;t,u} : \operatorname{Lie}(\mathbf{G})(E)_{x,t:u} \longrightarrow \mathbf{G}(E)_{x,t:u}\right)_{\substack{E/F \text{ finite and tamely ramified} \\ \mathbf{T} \text{ a tame maximal } E\text{-torus in } \mathbf{G} \\ x \in \mathcal{B}(\mathbf{T},E) \\ t,u \in \mathbb{R} \\ t \le u \le 2t}}$$

(in fact, of isomorphisms, by Proposition 1.6.2(a) of *loc. cit.*). By Corollary 1.6.6 of *loc. cit.*, given E, x, t, and u, the map $\varphi_{\mathbf{T}(E),x;t,u}$ does not depend on the choice of tame maximal E-torus $\mathbf{T}$ such that $x \in \mathcal{B}(\mathbf{T}, E)$; so we may write $\mathsf{e}^E_{x,t:u} = \varphi_{\mathbf{T}(E),x;t,u}$ for any such choice. These maps are the *Moy–Prasad isomorphisms*. It is easy to extend the definition of $\mathsf{e}^E_{x,t:u}$ to all $t, u \in \widetilde{\mathbb{R}}$ (not just $t, u \in \mathbb{R}$) satisfying $t \le u \le 2t$.

By construction, the resulting family satisfies Hypotheses A.1(1)–A.1(3). Hypothesis A.1(5) is just Proposition 1.6.2(b) of [1]. Hypothesis A.1(4) is Propositions 1.6.2(b) and 1.6.3 of *loc. cit.* By Remark A.3, it suffices to show Hypothesis A.1(6) in case $\mathbf{G}$ is E-split, in which case it follows (as in the proof of Proposition 1.9.2 of [1]) from Definition 5.13 of [6] and the construction of $\mathsf{e}^E_{x,t:u}$.

In [1, §1.5] one also finds a family of bijections

$$\left(\varphi_{T,x} : \mathfrak{g}_{x,0+} \longrightarrow G_{x,0+}\right)_{\substack{\mathbf{T} \text{ a tame maximal } F\text{-torus in } \mathbf{G} \\ x \in \mathcal{B}(\mathbf{T},F).}}$$

For fixed $\mathbf{T}$ and x, the bijection $\varphi_{T,x}$ is constructed (as in §1.3 of *loc. cit.*) from the various bijections $\mathsf{e}_{x,t:t+}$ after choosing representatives of the $\mathfrak{g}_{x,t+}$-cosets in $\mathfrak{g}_{x,t}$ and the $G_{x,t+}$-cosets in $G_{x,t}$ for $t \in \mathbb{R}$. Regardless of the choice of representatives, the construction ensures that the maps $\mathsf{e}_{\mathbf{T},x} := \varphi_{T,x}$ satisfy Hypothesis A.7(1); and it is observed in Remark 4.1.1 of [4] that they satisfy Hypothesis A.7(2). However, we must make our choices with some delicacy to ensure that Hypothesis A.7(3) is satisfied.

The choices are made as follows. Fix $t \in \mathbb{R}$. Note that, if $\left\{ \mathbf{H}^j \mid j \in J \right\}$ is a family of (not necessarily connected) reductive F-subgroups of $\mathbf{G}$ containing $\mathbf{T}$, then $\bigcap_{j \in J} \mathbf{H}_j$ is also (not necessarily connected) reductive; and, by Lemma 5.29 of [6],

$$\bigcap_{j \in J} H^j_{x,t} G_{x,t+} = \bigcap_{j \in J} (H^{j\circ}, G)_{x,(t,t+)} = \left(\left(\bigcap_{j \in J} H^j\right)^\circ, G\right)_{x,(t,t+)} \subseteq \left(\bigcap_{j \in J} H^j_{x,t}\right) G_{x,t+}.$$

Thus, if $g \in G_{x,t}$, then there exists an element $h \in gG_{x,t+}$ such that $h \in H$ whenever $\mathbf{H}$ is a reductive F-subgroup of $\mathbf{G}$ containing $\mathbf{T}$ such that $H \cap gG_{x,t+} \ne \emptyset$. We choose any such element h as the representative for the coset $gG_{x,t+}$. Similarly, we choose, for every $X \in \mathfrak{g}_{x,t}$, a coset

representative Y for $X + \mathfrak{g}_{x,t+}$ such that $Y \in \mathfrak{h}$ whenever $\mathbf{H}$ is a reductive F-subgroup of $\mathbf{G}$ containing $\mathbf{T}$ such that $\mathfrak{h} \cap X + \mathfrak{g}_{x,t+} \neq \emptyset$.

By Lemma A.4, if $t \in \mathbb{R}$, $X \in \mathfrak{g}_{x,t}$, and $g \in G_{x,t}$, then the sets of reductive F-subgroups $\mathbf{H}$ of $\mathbf{G}$, containing $\mathbf{T}$, such that $\mathfrak{h} \cap X + \mathfrak{g}_{x,t+} \neq \emptyset$, and such that $H \cap gG_{x,t+} \neq \emptyset$, are the same. Thus it follows from the construction that Hypothesis A.7(3) is satisfied for $\mathsf{e}_{\mathbf{T},x}$. $\qquad\square$

APPENDIX B. AN ORBITAL-INTEGRAL FORMULA

In this section, suppose that
- $\mathbf{G}$ is a reductive algebraic F-group (possibly disconnected), split over a tame extension of F, satisfying Hypotheses A.1 and A.7,
- $(x,r) \in \mathcal{B}(\mathbf{G}, F) \times \mathbb{R}_{>0}$,
- $X^* \in \mathfrak{g}^*_{x,-r} \smallsetminus \mathfrak{g}^*_{(-r)+}$ satisfies condition **GE1** of [65, §8],
- ϕ is a linear character of $G_{x,r:r+}$, and
- $\phi \circ \mathsf{e}_{x,r:r+} = \Lambda \circ X^*$.

Note that, by Proposition A.8, the requirement that $\mathbf{G}$ satisfy Hypothesis A.1 is superfluous. Put $\mathbf{G}' = C_{\mathbf{G}}(X^*)^\circ$. We choose, arbitrarily, a tame maximal F-torus $\mathbf{T}$ with $x \in \mathcal{B}(\mathbf{T}, F)$, and write $\mathsf{e}_x = \mathsf{e}_{\mathbf{T},x}$, where $\mathsf{e}_{\mathbf{T},x}$ is as in Hypothesis A.7. (The resulting map will depend on our choice of $\mathbf{T}$, but this dependence will not affect our proof.)

Lemma B.1. *Suppose that*
- *$d \in \mathbb{R}_{>0}$ with $d < r$,*
- *$h \in G_{x,d}$, and*
- *$[h, \hat{\phi}]$ is trivial on $G_{x,r-d}$.*

Then $h \in (G', G)_{x,(d,d+)}$.

Here, $[h, \hat{\phi}]$ is the character $g \mapsto \hat{\phi}([h^{-1}, g])$ of $G_{x,r-d}$.

Proof. Fix $X \in \mathfrak{g}_{x,r-d}$, and let g be any element of $\mathsf{e}_{x,(r-d):(r-d)+}(X)$. By Hypothesis A.1(4), $[h^{-1}, g]$ belongs to $\mathsf{e}_{x,r:r+}\big((\mathrm{Ad}(h)^{-1} - 1)X\big)$, so

$$
\begin{aligned}
1 = [h, \hat{\phi}](g) &= \hat{\phi}([h^{-1}, g]) \\
&= \Lambda\big(X^*\big((\mathrm{Ad}(h)^{-1} - 1)X\big)\big) = \Lambda\big((\mathrm{Ad}^*(h) - 1)X^*(X)\big).
\end{aligned}
$$

Since $X \in \mathfrak{g}_{x,r-d}$ was arbitrary, we have $(\mathrm{Ad}^*(h) - 1)X^* \in \mathfrak{g}^*_{x,(d-r)+}$. By Lemma 8.4 of [65] (the proof of which uses only condition **GE1**, not the full definition of genericity), $h \in (G', G)_{x,(d,d+)}$. $\qquad\square$

Since it is often easier to work with orbital integrals over connected groups than over disconnected ones, we present a basic result describing a G-orbital integral as a sum of G°-orbital integrals. Choose an invariant measure $d\dot{h}$ on $G/C_G(X^*)$. Fix $g \in G$. Then $d\dot{h}$ affords a natural choice

of invariant measure on $G/{}^gC_G(X^*) = G/C_G({}^gX^*)$. Since $G^\circ/C_{G^\circ}({}^gX^*)$ embeds naturally as an open subset of $G/C_G({}^gX^*)$, it inherits this invariant measure. For convenience, we will write again $d\dot{h}$ for the various measures occurring in this way. We will take all orbital integrals with respect to these measures. Notice that the induced measure on each $G/C_G({}^gX^*)G^\circ$ is just counting measure.

Lemma B.2. *We have that* $\hat{\mu}_{X^*}^G = [C_G(X^*) : C_{G^\circ}(X^*)]^{-1} \sum_{g \in G/G^\circ} \hat{\mu}_{{}^gX^*}^{G^\circ}$.

The statement may be interpreted as an equality of distributions, or of functions. We will prove the equality of distributions; that of functions follows immediately.

Proof. For $f \in C_c^\infty(\mathfrak{g})$, we have that

$$\hat{\mu}_{X^*}^G(f) = \int_{G/C_G(X^*)} \hat{f}({}^hX^*)d\dot{h}$$

$$= \sum_{g \in G/C_G(X^*)G^\circ} \int_{G^\circ/C_{G^\circ}(X^*)} \hat{f}({}^{gh}X^*)d\dot{h}$$

$$= [C_G(X^*) : C_{G^\circ}(X^*)]^{-1} \sum_{g \in G/G^\circ} \int_{G^\circ/C_{G^\circ}(X^*)} \hat{f}({}^{gh}X^*)d\dot{h}.$$

For fixed $g \in G$, the inner integral is just $\int_{G^\circ/C_{G^\circ}({}^gX^*)} \hat{f}({}^{hg}X^*)d\dot{h} = \hat{\mu}_{{}^gX^*}^{G^\circ}(f)$. The result follows. $\square$

Remark B.3. Our result as stated does not actually require that X^* satisfy condition **GE1** of [65, §8], only that the relevant orbital integrals converge. In particular, the result holds for any semisimple X^*.

Compare the next result to Lemma 6.3.5 of [4].

Lemma B.4. *Suppose that*
- $\mathbf{G}'/Z(\mathbf{G}')$ *is* F-*anisotropic,*
- Z *is a cocompact subgroup of* $Z(G^\circ)$ *that is normal in* G,
- $\mathcal{K}$ *is a compact open subgroup of* G°, *and*
- $Y \in \bigcup_{y \in \mathcal{B}(\mathbf{G}, F)} \mathfrak{g}_{y,r}$ *is regular semisimple.*

If $Y \in \mathfrak{g}_{x,r}$ *and* $\gamma = \mathsf{e}_x(Y)$, *then*

$$\hat{\mu}_{X^*}^G(Y) = \int_{G/Z} \int_{\mathcal{K}} [G_{x,r}]({}^{gk}\gamma)\hat{\phi}({}^{gk}\gamma)dk\,d\dot{g},$$

where $d\dot{g}$ *is a Haar measure on* G/Z *and* dk *is the Haar measure on* $\mathcal{K}$, *normalized so that* $\mathrm{meas}\,\mathcal{K} = 1$. *If* ${}^GY \cap \mathfrak{g}_{x,r} = \emptyset$, *then*

$$\hat{\mu}_{X^*}(Y) = 0.$$

When $Y \in \mathfrak{g}_{x,r}$, the notation of the lemma is meant to convey that the integrand equals $\hat{\phi}(^{gk}\gamma)$ if $^{gk}\gamma \in G_{x,r}$, and equals 0 otherwise.

Proof. Since $G_{x,r} = G_{x,r}^\circ$, and since, for $g \in G$, replacing X^* by $^gX^*$ has the effect of changing ϕ to ϕ^g, we see by Lemma B.2 that it suffices to prove this result in case G is connected.

The map $g \mapsto \min\left\{ d_x(^{k_1 g k_2}Y) \mid k_1 \in G_{x,0+}, k_2 \in \mathcal{K} \right\}$ on G is locally constant. For $t \in \mathbb{R} \cup \{\infty\}$, denote by $G(t)$ the preimage of t under this map, and put

$$I(t) = \int_{G(t)/Z} \int_{\mathcal{K}} \Lambda(X^*(^{gk}Y))dk\, d\dot{g}$$

and

$$I'(t) = \int_{G(t)/Z} \int_{\mathcal{K}} [\mathfrak{g}_{x,r}](^{gk}Y)\Lambda(X^*(^{gk}Y))dk\, d\dot{g}.$$

We show that $I(t) = I'(t)$ for $t \in \mathbb{R} \cup \{\infty\}$.

Fix $t \in \mathbb{R} \cup \{\infty\}$. If $t \geq r$, then the desired equality is obvious, so we may suppose that $t < r$. Denote by $t = t_0 < t_1 < \cdots < t_m$ the distinct values of d_x in $[t, r)$. Given a compact open subgroup $\mathcal{K}'$ of G such that $k'G(t) = G(t)$ for all $k' \in \mathcal{K}'$ (for example, an open subgroup of $G_{x,0+}$), we have

$$I(t) = [\mathcal{K}' : \mathcal{K}''] \int_{G(t)/Z} \int_{\mathcal{K}''} \int_{\mathcal{K}} \Lambda(X^*(^{k'gk}Y))dk\, dk'\, d\dot{g}$$

$$= \int_{G(t)/Z} \sum_{h \in \mathcal{K}''\backslash\mathcal{K}'} \int_{\mathcal{K}''} \int_{\mathcal{K}} \Lambda(X^*(^{k'hgk}Y))dk\, dk'\, d\dot{g}$$

$$= \int_{G(t)/Z} \int_{\mathcal{K}'} \int_{\mathcal{K}} \Lambda(X^*(^{k'gk}Y))dk\, dk'\, d\dot{g}$$

$$= \int_{G(t)/Z} \int_{\mathcal{K}} \int_{\mathcal{K}'} \Lambda(X^*(^{k'gk}Y))dk'\, dk\, d\dot{g},$$

where $\mathcal{K}'' = \mathcal{K}' \cap G_{x,(r-t)+}$ and dk' is the Haar measure on $\mathcal{K}'$, normalized so that $\mathrm{meas}(\mathcal{K}') = 1$. An easy generalization allows us to handle several subgroups $\mathcal{K}'_0, \ldots, \mathcal{K}'_m$ as above, so we obtain

$(**)$

$$I(t) = \int_{G(t)/Z} \int_{\mathcal{K}} \int_{G_{x,r-t_m}} \cdots \int_{G_{x,r-t_0}} \Lambda(X^*(^{h_m \cdots h_0 gk}Y))dh_0 \cdots dh_m\, dk\, d\dot{g},$$

where, for $j = 0, \ldots, m$, dh_j is the Haar measure on $G_{x,r-t_j}$ normalized so that $\mathrm{meas}(G_{x,r-t_j}) = 1$. Now fix $g \in G(t)$ and $k \in \mathcal{K}$, and put $\widetilde{Y} = {}^{gk}Y$. Then either $[\mathfrak{g}_{x,r}](\widetilde{Y}) = 1$, or there is $0 \leq j \leq m$ such that

$\mathrm{d}_x(\widetilde{Y}) = t_j$ (in particular, $\widetilde{Y} \notin \mathfrak{g}_{x,t_j+}$). In the latter case, fix $h_i \in G_{x,r-t_i}$ for $0 \le i \le m$. By Remark A.2, we have that $^{h_j}\widetilde{Y} - \widetilde{Y} \in \mathfrak{g}_{x,r}$ and (since $h_{j-1} \cdots h_0 \in G_{x,r-t_{j-1}} \subseteq G_{x,(r-t_j)+}$) also that $^{h_{j-1}\cdots h_0}\widetilde{Y} - \widetilde{Y} \in \mathfrak{g}_{x,r+}$, hence that $^{h_j}\big(^{h_{j-1}\cdots h_0}\widetilde{Y} - \widetilde{Y}\big) \in \mathfrak{g}_{x,r+}$. Thus, by another application of Remark A.2,

$$^{h_m\cdots h_0}\widetilde{Y} - {}^{h_m\cdots h_{j+1}}\widetilde{Y}$$
$$= {}^{h_m\cdots h_{j+1}}\Big(^{h_j}\big(^{h_{j-1}\cdots h_0}\widetilde{Y} - \widetilde{Y}\big) + {}^{h_j}\widetilde{Y} - \widetilde{Y}\Big)$$
$$\in {}^{h_j}\widetilde{Y} - \widetilde{Y} + \mathfrak{g}_{x,r+}.$$

That is,

$$\Lambda(X^*(^{h_m\cdots h_0}\widetilde{Y})) = \Lambda(X^*(^{h_m\cdots h_{j+1}}\widetilde{Y}))\Lambda(X^*(^{h_j}\widetilde{Y} - \widetilde{Y})),$$

so the inner integrals in $(**)$ are a constant multiple of

$$(***) \qquad\qquad \int_{G_{x,r-t_j}} \Lambda(X^*(^{h_j}\widetilde{Y} - \widetilde{Y}))dh_j.$$

Now, for $h_j, h'_j \in G_{x,r-t_j}$, we have by Remark A.2 that $^{h'_j}\widetilde{Y} - \widetilde{Y} \in \mathfrak{g}_{x,r}$, so

$$^{h_j h'_j}\widetilde{Y} - \widetilde{Y} = {}^{h_j}\big(^{h'_j}\widetilde{Y} - \widetilde{Y}\big) + \big(^{h_j}\widetilde{Y} - \widetilde{Y}\big)$$
$$\in \big(^{h'_j}\widetilde{Y} - \widetilde{Y}\big) + \big(^{h_j}\widetilde{Y} - \widetilde{Y}\big) + \mathfrak{g}_{x,r+}.$$

That is, $\varphi_{\widetilde{Y}} : h_j \mapsto \Lambda(X^*(^{h_j}\widetilde{Y} - \widetilde{Y}))$ is a homomorphism on $G_{x,r-t_j}$; so $(***)$ equals 0 unless $\varphi_{\widetilde{Y}}$ is trivial there.

Suppose that it is trivial. Choose $\tilde{\gamma} \in \mathsf{e}_{x,t_j:t_j+}(\widetilde{Y})$. By Hypothesis A.1(4), $[h_j, \tilde{\gamma}] \in \mathsf{e}_{x,r:r+}((\mathrm{Ad}(h_j) - 1)\widetilde{Y})$, so

$$[\tilde{\gamma}^{-1}, \hat{\phi}](h_j) = \hat{\phi}([h_j, \tilde{\gamma}])^{-1} = \Lambda(X^*((\mathrm{Ad}(h_j) - 1)\widetilde{Y})^{-1} = 1,$$

for $h_j \in G_{x,r-t_j}$. That is, $[\tilde{\gamma}^{-1}, \hat{\phi}]$ is trivial on $G_{x,r-t_j}$; so, by Lemma B.1, $\tilde{\gamma} \in (G', G)_{x,(t_j, t_j+)}$. Then Hypothesis A.1(6) gives $\widetilde{Y} \in (\mathfrak{g}', \mathfrak{g})_{x,(t_j, t_j+)}$. On the other hand, $\widetilde{Y} \in \mathfrak{g}_r \subseteq \mathfrak{g}_{t_j+}$. Therefore, by Corollary 3.7.8 of [2], we have that $\widetilde{Y} \in \mathfrak{g}'_{t_j+} + \mathfrak{g}_{x,t_j+}$. Since $\mathbf{G}'/Z(\mathbf{G}')$ is F-anisotropic, we have that $\mathfrak{g}'_{t_j+} = \mathfrak{g}'_{x,t_j+} \subseteq \mathfrak{g}_{x,t_j+}$, so in fact $\widetilde{Y} \in \mathfrak{g}_{x,t_j+}$, a contradiction.

We have shown that

$$\int_{G_{x,r-t_m}} \cdots \int_{G_{x,r-t_0}} \Lambda(X^*(^{h_m\cdots h_0}\widetilde{Y}))dh_0 \cdots dh_m$$
$$= (\text{const}) \int_{G_{x,r-t_j}} \Lambda(X^*(^{h_j}\widetilde{Y} - \widetilde{Y}))dh_j = 0$$

whenever $\widetilde{Y} \notin \mathfrak{g}_{x,r}$. Thus $I(t) = I'(t)$. By a theorem of Huntsinger (see [4, Appendix A]), $\hat{\mu}_{X^*}(Y) = \sum_{t \in \mathbb{R} \cup \{\infty\}} I(t)$; so $\hat{\mu}_{X^*}(Y) = \sum_{t \in \mathbb{R} \cup \{\infty\}} I'(t)$. In particular, if ${}^G Y \cap \mathfrak{g}_{x,r} = \emptyset$, then $\hat{\mu}_{X^*}(Y) = 0$.

Now suppose that $Y \in \mathfrak{g}_{x,r}$. By Hypothesis A.7(2), ${}^{gk}\gamma \in G_{x,r}$ if and only if ${}^{gk}Y \in \mathfrak{g}_{x,r}$, in which case ${}^{gk}\gamma$ belongs to $e_{x,r:r+}({}^{gk}Y)$. Recall that $\hat{\phi} \circ e_{x,r:r+} = \Lambda \circ X^*$ on $\mathfrak{g}_{x,r:r+}$. Thus

$$\int_{G/Z} \int_{\mathcal{K}} [G_{x,r}]({}^{gk}\gamma)\hat{\phi}({}^{gk}\gamma)dk\,d\dot{g}$$

$$= \int_{G/Z} \int_{\mathcal{K}} [\mathfrak{g}_{x,r}]({}^{gk}Y)\Lambda(X^*({}^{gk}Y))dk\,d\dot{g}$$

$$= \sum_{t \in \mathbb{R}} I'(t) = \hat{\mu}_{X^*}(Y). \quad \square$$

INDEX OF NOTATION

References

[1] Jeffrey D. Adler, *Refined anisotropic K-types and supercuspidal representations*, Pacific J. Math. **185** (1998), no. 1, 1–32. MR **1653184 (2000f:**22019)

[2] Jeffrey D. Adler and Stephen DeBacker, *Some applications of Bruhat–Tits theory to harmonic analysis on the Lie algebra of a reductive p-adic group*, with appendices by Reid Huntsinger and Gopal Prasad, Michigan Math. J. **50** (2002), no. 2, 263–286. MR **1914065 (2003g:**22016)

[3] Jeffrey D. Adler, Lawrence Corwin, and Paul J. Sally Jr., *Discrete series characters of division algebras and GL_n over a p-adic field*, Contributions to automorphic forms, geometry, and number theory (H. Hida, D. Ramakrishnan, and F. Shahidi, eds.), Johns Hopkins University Press, Baltimore, MD, 2004, pp. 57–64. MR **2058603 (2005d:**22015)

[4] Jeffrey D. Adler and Stephen DeBacker, *Murnaghan–Kirillov theory for supercuspidal representations of tame general linear groups*, J. Reine Angew. Math. **575** (2004), 1–35. MR **2097545 (2005j:**22008)

[5] Jeffrey D. Adler and Jonathan Korman, *The local character expansion near a tame, semisimple element*, Amer. J. Math. **129** (2007), no. 2, 381–403, available at `arXiv:math.RT/0503051`.

[6] Jeffrey D. Adler and Loren Spice, *Good product expansions for tame elements of p-adic groups*, Int. Math. Res. Pap. **2008**, DOI 10.1093/imrp/rpn003, available at `arXiv:math.RT/0611554`.

[7] Magdy Assem, *The Fourier transform and some character formulae for p-adic SL_l, l a prime*, Amer. J. Math. **116** (1994), no. 6, 1433–1467. MR **1305872 (96i:**22037)

[8] John Boller, *Characters of some supercuspidal representations of p-adic $\mathrm{Sp}_4(F)$*, Ph. D. Thesis, The University of Chicago, 1999.

[9] Colin J. Bushnell and Philip C. Kutzko, *The admissible dual of $\mathrm{GL}(N)$ via compact open subgroups*, Annals of Mathematics Studies, vol. 129, Princeton University Press, Princeton, NJ, 1993. MR **1204652 (94h:**22007)

[10] _____, *The admissible dual of $\mathrm{SL}(N)$. I*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 2, 261–280. MR **1209709 (94a:**22033)

[11] _____, *The admissible dual of $\mathrm{SL}(N)$. II*, Proc. London Math. Soc. (3) **68** (1994), no. 2, 317–379. MR **1253507 (94k:**22035)

[12] Colin J. Bushnell and Guy Henniart, *Local tame lifting for $\mathrm{GL}(N)$. I. Simple characters*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 105–233. MR **1423022 (98m:**11129)

[13] William Casselman, *Characters and Jacquet modules*, Math. Ann. **230** (1977), no. 2, 101–105. MR 0492083 (58 #11237)

[14] Lawrence Corwin, *Representations of division algebras over local fields*, Advances in Math. **13** (1974), 259–267. MR 0347780 (50 #281)

[15] Lawrence Corwin and Roger E. Howe, *Computing characters of tamely ramified p-adic division algebras*, Pacific J. Math. **73** (1977), no. 2, 461–477. MR 0492084 (58 #11238)

[16] Lawrence Corwin, Allen Moy, and Paul J. Sally Jr., *Degrees and formal degrees for division algebras and GL_n over a p-adic field*, Pacific J. Math. **141** (1990), no. 1, 21–45. MR **1028263 (90k:**22025)

[17] _____, *Supercuspidal character formulas for GL_ℓ*, Representation theory and harmonic analysis (Cincinnati, OH, 1994), 1995, pp. 1–11. MR **1365530 (96m:**22037)

[18] Clifton Cunningham, *Characters of depth-zero, supercuspidal representations of the rank-2 symplectic group*, Canad. J. Math. **52** (2000), no. 2, 306–331. MR **1755780 (2001f:**22055)

[19] Stephen DeBacker, *On supercuspidal characters of* GL_ℓ, ℓ *a prime*, Ph. D. Thesis, The University of Chicago, 1997.

[20] Stephen DeBacker and Paul J. Sally Jr., *Germs, characters, and the Fourier transforms of nilpotent orbits*, The mathematical legacy of Harish-Chandra (Robert S. Doran and V. S. Varadarajan, eds.), Proceedings of Symposia in Pure Mathematics, vol. 68, American Mathematical Society, Providence, RI, 2000, pp. 191–221. MR **1767897 (2001i:**22022)

[21] Stephen DeBacker, *Homogeneity results for invariant distributions of a reductive p-adic group*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 3, 391–422 (English, with English and French summaries). MR **1914003 (2003i:**22019)

[22] Stephen DeBacker and Mark Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. Math., to appear.

[23] Stephen DeBacker and David Kazhdan, *Murnaghan-Kirillov theory for depth zero supercuspidal representations: reduction to cuspidal local systems* (April 7, 2006), preprint.

[24] Pierre Deligne, *Le support du caractère d'une représentation supercuspidale*, C. R. Acad. Sci. Paris Sér. A-B **283** (1976), no. 4, Aii, A155–A157 (French, with English summary). MR 0425033 (54 #12991)

[25] Paul Gérardin, *Weil representations associated to finite fields*, J. Algebra **46** (1977), no. 1, 54–101. MR 0460477 (57 #470)

[26] Jeffrey Hakim and Fiona Murnaghan, *Tame supercuspidal representations of* $\mathrm{GL}(n)$ *distinguished by a unitary group*, Compositio Math. **133** (2002), no. 2, 199–244. MR **1923582 (2003g:**22019)

[27] Thomas C. Hales, *Hyperelliptic curves and harmonic analysis (why harmonic analysis on reductive p-adic groups is not elementary)*, Representation theory and analysis on homogeneous spaces (Simon Gindikin, Frederick P. Greenleaf, and Paul J. Sally Jr., eds.), Contemporary Mathematics, vol. 177, American Mathematical Society, Providence, RI, 1994, pp. 137–169. MR **1303604 (96d:**22024)

[28] Harish-Chandra, *Harmonic analysis on reductive p-adic groups*, notes by G. van Dijk, Springer–Verlag, Berlin, 1970. MR 0414797 (54 #2889)

[29] _____, *A submersion principle and its applications*, Geometry and analysis: Papers dedicated to the memory of V. K. Patodi, Indian Academy of Sciences, Bangalore, 1980, pp. 95–102. MR **592255 (82e:**22032)

[30] _____, *Admissible invariant distributions on reductive p-adic groups*, with a preface and notes by Stephen DeBacker and Paul J. Sally, Jr., University Lecture Series, vol. 16, American Mathematical Society, Providence, RI, 1999. MR **1702257 (2001b:**22015)

[31] Guy Henniart, *Correspondance de Jacquet-Langlands explicite. I. Le cas modéré de degré premier*, Séminaire de Théorie des Nombres, Paris, 1990–91, 1993, pp. 85–114 (French). MR **1263525 (95d:**11064)

[32] Roger E. Howe, *On the character of Weil's representation*, Trans. Amer. Math. Soc. **177** (1973), 287–298. MR 0316633 (47 #5180)

[33] _____, *The Fourier transform and germs of characters (case of* Gl_n *over a p-adic field)*, Math. Ann. **208** (1974), 305–322. MR 0342645 (49 #7391)

[34] _____, *Tamely ramified supercuspidal representations of* Gl_n, Pacific J. Math. **73** (1977), no. 2, 437–460. MR 0492087 (58 #11241)

[35] _____, *Some qualitative results on the representation theory of* Gl_n *over a p-adic field*, Pacific J. Math. **73** (1977), no. 2, 479–538. MR 0492088 (58 #11242)

[36] Nathan Jacobson, *Basic algebra. I*, W. H. Freeman and Company, New York, 1985. MR **780184 (86d:**00001)

[37] Ju-Lee Kim and Fiona Murnaghan, *Character expansions and unrefined minimal K-types*, Amer. J. Math. **125** (2003), no. 6, 1199–1234. MR **2018660 (2004k:**22024)

[38] _____, *K-types and* Γ-*asymptotic expansions*, J. Reine Angew. Math. **592** (2006), 189-236.

[39] Ju-Lee Kim, *Supercuspidal representations: an exhaustion theorem*, J. Amer. Math. Soc. **20** (2007), no. 2, 273–320 (electronic). MR 2276772

[40] Philip C. Kutzko, *On the supercuspidal representations of* Gl_2, Amer. J. Math. **100** (1978), no. 1, 43–60. MR 0507253 (58 #22411a)

[41] _____, *On the supercuspidal representations of* Gl_2. *II*, Amer. J. Math. **100** (1978), no. 4, 705–716. MR 0507254 (58 #22411b)

[42] Erasmus Landvogt, *Some functorial properties of the Bruhat–Tits building*, J. Reine Angew. Math. **518** (2000), 213–241. MR **1739403 (2001g:**20029)

[43] Rudolf Lidl and Harald Niederreiter, *Finite fields*, Encyclopedia of Mathematics and its Applications, vol. 20, Cambridge University Press, Cambridge, 1997. MR **1429394 (97i:**11115)

[44] Allen Moy, *Local constants and the tame Langlands correspondence*, Amer. J. Math. **108** (1986), no. 4, 863–930. MR **853218 (88b:**11081)

[45] Allen Moy and Gopal Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), no. 1–3, 393–408. MR **1253198 (95f:**22023)

[46] _____, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), no. 1, 98–121. MR **1371680 (97c:**22021)

[47] Fiona Murnaghan, *Local character expansions for supercuspidal representations of* $\mathrm{U}(3)$, Canad. J. Math. **47** (1995), no. 3, 606–640. MR **1346155 (96i:**22026)

[48] _____, *Characters of supercuspidal representations of* $\mathrm{SL}(n)$, Pacific J. Math. **170** (1995), no. 1, 217–235. MR **1359978 (96k:**22030)

[49] _____, *Characters of supercuspidal representations of classical groups*, Ann. Sci. École Norm. Sup. (4) **29** (1996), no. 1, 49–105. MR **1368705 (98c:**22016)

[50] _____, *Local character expansions and Shalika germs for* $\mathrm{GL}(n)$, Math. Ann. **304** (1996), no. 3, 423–455. MR **1375619 (98b:**22020)

[51] Cary Rader and Allan Silberger, *Some consequences of Harish-Chandra's submersion principle*, Proc. Amer. Math. Soc. **118** (1993), no. 4, 1271–1279. MR **1169888 (93j:**22032)

[52] Paul J. Sally Jr., *Unitary and uniformly bounded representations of the two by two unimodular group over local fields*, Amer. J. Math. **90** (1968), 406–443. MR 0229755 (37 #5329)

[53] Paul J. Sally Jr. and Joseph A. Shalika, *Characters of the discrete series of representations of* $\mathrm{SL}(2)$ *over a local field*, Proc. Nat. Acad. Sci. U.S.A. **61** (1968), 1231–1237. MR 0237713 (38 #5994)

[54] _____, *The Plancherel formula for* $\mathrm{SL}(2)$ *over a local field*, Proc. Nat. Acad. Sci. U.S.A. **63** (1969), 661–667. MR 0364559 (51 #813)

[55] _____, *The Fourier transform of orbital integrals on* SL_2 *over a p-adic field*, Lie group representations. II (R. Herb, S. Kudla, R. Lipsman, and J. Rosenberg, eds.),

Lecture Notes in Mathematics, vol. 1041, Springer–Verlag, Berlin, 1984, pp. 303–340.

[56] Paul J. Sally Jr., *Some remarks on discrete series characters for reductive p-adic groups*, Representations of Lie groups, Kyoto, Hiroshima, 1986, 1988, pp. 337–348. MR **1039842 (91g:**22026)

[57] Vincent Sécherre, *Représentations lisses de* $\mathrm{GL}_m(D)$. *III. Types simples*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 6, 951–977 (French, with English and French summaries). MR **2216835 (2007a:**22010)

[58] Jean-Pierre Serre, *Galois cohomology*, Springer Monographs in Mathematics, Springer–Verlag, Berlin, 2002 (English). MR **1867431 (2002i:**12004)

[59] Hideo Shimizu, *Some examples of new forms*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), no. 1, 97–113. MR 0447121 (56 #5436)

[60] Allan J. Silberger, PGL_2 *over the p-adics: its representations, spherical functions, and Fourier analysis*, Lecture Notes in Mathematics, Vol. 166, Springer–Verlag, Berlin, 1970. MR 0285673 (44 #2891)

[61] Loren Spice, *Supercuspidal characters of* SL_ℓ *over a p-adic field, ℓ a prime*, Amer. J. Math. **127** (2005), no. 1, 51–100. MR **2115661 (2005k:**22028)

[62] _____, *Topological Jordan decompositions*, J. Algebra **319** (2008), 3141–3163, available at `arXiv:math.GR/0612475`.

[63] Shaun Stevens, *The supercuspidal representations of p-adic classical groups*, Inventiones Math. **172** (2008), 289–352, available at `arXiv:math.RT/0607622`.

[64] Jean-Loup Waldspurger, *Une formule des traces locale pour les algèbres de Lie p-adiques*, J. Reine Angew. Math. **465** (1995), 41–99 (French). MR **1344131 (96i:**22039)

[65] Jiu-Kang Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), no. 3, 579–622 (electronic). MR **1824988 (2002f:**22033)

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