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Exact Multivariate Integration on Simplices: an Explanation of The Lasserre-Avrachenkov Theorem

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EXACT MULTIVARIATE INTEGRATION ON SIMPLICES: AN EXPLANATION OF THE LASSERRE-AVRACHENKOV THEOREM

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Abstract. Because the traditional method for evaluating integrals over higher dimensional simplices can be computationally challenging, Lasserre and Avrachenkov established an equation for evaluating integrals of symmetric multilinear forms over simplices. Before an integral can be evaluated in this manner the starting homogeneous polynomial must be expressed as a symmetric multilinear form, by way of a polarization identity. In this paper, the Lasserre-Avrachenkov method for evaluating integrals over simplices is explained and explored, beginning with a homogenous polynomial and a simplex, and ending with an exact value. This method can be used in computer programs that provide an efficient method for precisely evaluating integrals over simplices in higher dimensions.

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1. Introduction

The evaluation of integrals over simplices has many applications particularly in the field of probability. With these integrals playing a part in larger problems, we need to be able to quickly and efficiently evaluate these integrals. However, as the dimensions increase the evaluation of these integrals by the traditional method becomes significantly more computationally complex. When working in dimensions greater than three, where visualization become impossible, simply determining the parametrization and bounds to break up the integral into evaluable parts becomes extraordinarily difficult. In addition, writing a program to parametrize and evaluate these integrals is equally challenging. Consequently, there was a need for a simpler formula.

In their paper "The Multi-Dimensional Version of $\int_a^b x^p dx$" [4] Jean B. Lasserre and Konstantin E. Avrachenkov proved an exact formula for evaluating integrals of polynomials over simplices in n dimensions. Essentially, this formula states that the integral over a simplex of a homogenous polynomial written as a symmetric multilinear form is equal to the sum of the polynomial at specified points times a constant. The constant is determined by the volume of the simplex, the number of dimensions, and the number of entries in the symmetric multilinear form. As Lasserre and Avrachenkov describe, this formula is extremely useful because depends only on the $n+1$ vertices of the simplex. In addition, it applies to any general simplex and is not limited to just the canonical simplex. It is important to note when using this equation that the polynomial H over which the integral is evaluated must be a symmetric multilinear form. However, it is possible to write any homogenous polynomial as a symmetric multilinear form by way of a polarization identity. Since any polynomial can be broken up into homogenous polynomials, the Lasserre-Avrachenkov Theorem provides a method for calculating the exact integral of any polynomial over a general simplex in n dimensions.

2. Preliminaries

Before we can proceed to the proof, it is important to explore some key terms and features of the theorem. Here, we will examine the concepts of simplices, symmetric multilinear forms, and the polarization identities.

Very generally, simplices are shapes in n dimensions with $n+1$ vertices. A simplex is most easily thought of in two dimensions, where a 2-simplex is a triangle, or in three dimensions, where a 3-simplex is a tetrahedron. Also note that the 0-simplex is considered to be a point and the 1-simplex is a line segment. As we move into higher dimensions, the same principles apply; however, the shape becomes impossible to visualize. A simplex is not necessarily regular or uniform in any way; however, it can be in specific cases. For example, the canonical simplex or unit simplex is a special uniform simplex in n dimensions which will be discussed more later. More formally, a simplex can be defined based on the idea of convex combinations. Note that all vectors used here will be column vectors. Let $\mathbf{x}$ be any generic point in a simplex and let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ be the vertices of an n dimensional simplex, then for all $\mathbf{x} \in \triangle_n$, $\mathbf{x} = \sum_{i=0}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. Note that $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ are called the barycentric coordinates of $\mathbf{x}$ with respect to $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$.

A multilinear form, or q-linear is a specific type of polynomial that is linear in each of its q variables separately. The variables in a multilinear form are all column vectors of length n where n is the dimension of the space. More specifically, this means that a multilinear form is a map $H:(\mathbb{R}^n)^q\to\mathbb{R}$ such that for any $i\in\{1,\ldots,q\}$ and $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}^n$

$$H(\mathbf{x}_1,\ldots,\mathbf{x}_{i-1},c\mathbf{x}_i,\mathbf{x}_{i+1},\ldots,x_q) = cH(\mathbf{x}_1,\ldots,\mathbf{x}_q) \text{ and}$$

$$H(\mathbf{x}_1,\ldots,\mathbf{x}_{i-1},\mathbf{y}+\mathbf{z},\mathbf{x}_{i+1},\ldots,\mathbf{x}_q)= \\ H(\mathbf{x}_1,\ldots,\mathbf{x}_{i-1},\mathbf{y},\mathbf{x}_{i+1},\ldots,\mathbf{x}_q)+H(\mathbf{x}_1,\ldots,\mathbf{x}_{i-1},\mathbf{y}+\mathbf{z},\mathbf{x}_{i+1},\ldots,\mathbf{x}_q)$$

A multilinear form can also be symmetric, meaning that the order of the entries does not matter. Essentially, for a symmetric q-linear form, any permutation of the entries will yield the same result. This can be expressed more formally as $H(\mathbf{x}_1,\ldots,\mathbf{x}_q)=H(\mathbf{x}_{\pi(1)},\ldots,\mathbf{x}_{\pi(q)})$where $\pi:\{1,\ldots,q\}\to\{1,\ldots,q\}$ is a one-to-one and onto function representing any permutation.

Now that we know what a symmetric multilinear form is, it is important to know how to obtain one. Any homogenous polynomial can be written as a unique symmetric multilinear form by one of two polarization identities. Although the polarization identities differ slightly in appearance, they both yield the same results. In both cases, f is a homogenous polynomial of degree q and it is being rewritten in the form of the symmetric multilinear form $H(\mathbf{x}_1,\ldots,\mathbf{x}_q)$. The first polarization identity which is stated in [5] (eq. 4) gives that

$$H(\mathbf{x}_1,\ldots,\mathbf{x}_q)=\frac{1}{q!}\frac{\partial}{\partial\lambda_1}\cdots\frac{\partial}{\partial\lambda_q}f(\lambda_1\mathbf{x}_1+\ldots+\lambda_q\mathbf{x}_q)\,|_{\lambda=0}.$$

Alternatively, the second form of the polarization is given in [2](eq. 17) and [1] states that

$$H(\mathbf{x}_1,\ldots,\mathbf{x}_q)=\frac{1}{2^q q!}\sum_{\epsilon\in\pm1^q}\epsilon_1\epsilon_2\cdots\epsilon_q\, f\left(\sum_{i=1}^q\epsilon_i\mathbf{x}_i\right)$$

Examples of how to use both of these polarization identities to re-write a homogenous polynomial as a symmetric multilinear form can be found in the Examples section of this paper.

3. Proofs

The following theorem is from Lasserre and Avrachenkov 2.1 [4] and the proof below was adapted from the proof there.

Theorem. *1 Let $\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_n$ be the vertices of an n-dimensional simplex $\triangle_n$. Then, for a symmetric multilinear form $H:(\mathbb{R}^n)^q\to\mathbb{R}$, one has*

$$\int_{\triangle_n}H(\mathbf{x},\mathbf{x},\ldots,\mathbf{x})\,dx = \frac{vol(\triangle_n)}{\binom{n+q}{q}}\left[\sum_{0\le i_1\le i_2,\ldots,\le i_q\le n}H(\mathbf{x}_{i_1},\mathbf{x}_{i_2},\ldots,\mathbf{x}_{i_q})\right].$$

Proof. We will begin with the canonical simplex and then generalize. The canonical simplex, Ω_n, is defined such that for all $\mathbf{x}$ in $\mathbb{R}^n$ (where $\mathbf{x}$ is any point on the simplex) and all $k=1,2,\ldots,n$, $0\le\mathbf{x}_k\le1$ and $\sum\mathbf{x}_k\le1$. Equivalently the vertices of Ω_n are $(0,0,\ldots,0),e_1,e_2,\ldots,e_n$where $e_j=j^{\text{th}}$ standard basis vector. Essentially, the components of any point on the canonical simplex must be between zero and one. We will start with the formula for integrating a homogenous polynomial over the canonical simplex. The formula states that for the canonical simplex,

$$\int_{\Omega_n}\mathbf{x}_1^{\alpha_1}\cdots\mathbf{x}_n^{\alpha_n}\,dx = \frac{\alpha_1!\cdots\alpha_n!}{(n+\sum_i\alpha_i)!}. \tag{3.1}$$

Since we are working with a symmetric q-linear form, we can now use properties of this form to move from the canonical simplex to a general n-dimensional simplex, $\triangle_n$. In a general simplex $\triangle_n$ we can express any point within the simplex as a convex combination of the vertices, $\mathbf{x}_0,\ \mathbf{x}_1, \ldots, \mathbf{x}_n$. Therefore, any point on a simplex can be expressed as a linear combination of the vertices with weights greater than or equal to zero and weights that sum to one. In other words, for all $\mathbf{x} \in \triangle_n$, $\mathbf{x} = \sum_{i=0}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. Since $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$, we can write

$$\lambda_0 = 1 - \sum_{i=1}^{n} \lambda_i.$$

Also, since we have previously defined that $\sum_i \lambda_i = 1$, it must be the case that $\sum_{i=1}^{n} \lambda_i \leq 1$. Therefore, we are able the express the weights $(\lambda_1, \ldots, \lambda_n)$ as a general element (not necessarily a vertex) of the canonical simplex $\Omega_n \in \mathbb{R}^n$ where

$$\Omega := \left\{ \lambda \in \mathbb{R}^n | \sum_{i=1}^{n} \lambda_i \leq 1,\ \lambda_i \geq 0,\ i = 1,\ 2, \ldots,\ n \right\}.$$

Using this, we can rewrite $\mathbf{x} = \sum_{i=0}^{n} \lambda_i \mathbf{x}_i$ as

$$\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i + \left(1 - \sum_{i=1}^{n} \lambda_i\right) \mathbf{x}_0 \tag{3.2}$$

where $(\lambda_1, \ldots, \lambda_n) \in \Omega_n$. Using algebra on (3.2) we obtain,

$$\begin{aligned} \mathbf{x} &= \sum_{i=1}^{n} \lambda_i \mathbf{x}_i + \left(1 - \sum_{i=1}^{n} \lambda_i\right) \mathbf{x}_0 \\ &= \sum_{i=1}^{n} \lambda_i \mathbf{x}_i + \mathbf{x}_0 - \sum_{i=1}^{n} \lambda_i \mathbf{x}_0 \qquad (3.3) \\ &= \mathbf{x}_0 + \sum_{i=1}^{n} \lambda_i (\mathbf{x}_i - \mathbf{x}_0). \end{aligned}$$

We will now use a change of variables $x \to \lambda$ to rewrite (3.1). We begin by finding the Jacobian matrix,

$$J = \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \lambda_1} & \frac{\partial \mathbf{x}}{\partial \lambda_\mathbf{2}} & \cdots & \frac{\partial \mathbf{x}}{\partial \lambda_n} \end{bmatrix}$$

Using Calculus we obtain,

$$J = \begin{bmatrix} (\mathbf{x_1} - \mathbf{x}_0) & (\mathbf{x}_2 - \mathbf{x_0}) & \cdots & (\mathbf{x}_n - \mathbf{x}_0) \end{bmatrix}$$

Note that each of the entries, $(\mathbf{x_1} - \mathbf{x}_0)$, $(\mathbf{x}_2 - \mathbf{x}_0), \ldots$ $(\mathbf{x}_n - \mathbf{x}_0)$ contains n terms so J is an $n \times n$ matrix. Therefore, we can write,

$$\det(J) = \det(\mathbf{x}_1 - \mathbf{x}_0,\ \mathbf{x}_2 - \mathbf{x}_0,\ \ldots,\ \mathbf{x}_n - \mathbf{x}_0).$$

By a change of variables from $\mathbf{x} \to \lambda$, we obtain

$$\int_{\triangle_n} H(\mathbf{x},\ \mathbf{x},\ \ldots,\ \mathbf{x})\, dx \ = |\det(\mathbf{x}_1 - \mathbf{x}_0,\ \mathbf{x}_2 - \mathbf{x}_0,\ \ldots,\ \mathbf{x}_n - \mathbf{x}_0)| \times$$

$$\int_{\Omega_n} H\left(\sum_{i=1}^{n} \lambda_i \mathbf{x}_i + \left(1 - \sum_{i=1}^{n} \lambda_i\right) \mathbf{x}_0,\ \ldots,\ \sum_{i=1}^{n} \lambda_i \mathbf{x}_i + \left(1 - \sum_{i=1}^{n} \lambda_i\right) \mathbf{x}_0\right) d\lambda. \tag{3.4}$$

Based on the properties of a simplex, we know that $vol(\triangle_n) = \frac{1}{n!}|\det(\mathbf{x}_1 - \mathbf{x}_0,\, \mathbf{x}_2 - \mathbf{x}_0,\, \ldots,\, \mathbf{x}_n - \mathbf{x}_0)|$. Therefore, $|\det(\mathbf{x}_1 - \mathbf{x}_0,\, \mathbf{x}_2 - \mathbf{x}_0,\, \ldots,\, \mathbf{x}_n - \mathbf{x}_0)| = n!\, vol(\triangle_n)$. Plugging this into (3.4) we obtain,

$$\int_{\triangle_n} H(\mathbf{x},\, \mathbf{x},\, \ldots,\, \mathbf{x})\, d\mathbf{x} = n!\, vol(\triangle_n)\times$$
$$\int_{\Omega_n} H\left(\sum_{i=1}^{n}\lambda_i\mathbf{x}_i + \left(1-\sum_{i=1}^{n}\lambda_i\right)\mathbf{x}_0,\, \ldots,\, \sum_{i=1}^{n}\lambda_i\mathbf{x}_i + \left(1-\sum_{i=1}^{n}\lambda_i\right)\mathbf{x}_0\right)\, d\lambda$$

Expanding this expression, we obtain

$$\int_{\triangle_n} H(\mathbf{x},\, \mathbf{x},\, \ldots,\, \mathbf{x})\, d\mathbf{x} = n!\, vol(\triangle_n)\times \tag{3.5}$$
$$\int_{\Omega_n} H\{\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_n\mathbf{x}_n + \left(1-\sum_{i=1}^{n}\lambda_i\right)\mathbf{x}_0,\, \ldots,$$
$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_n\mathbf{x}_n + \left(1-\sum_{i=1}^{n}\lambda_i\right)\mathbf{x}_0\}d\lambda.$$

Since H is defined to be in symmetric multilinear form, we can now use the properties of a symmetric multilinear form to expand this expression. Using the property that $G(\mathbf{x}_1,\mathbf{x}_2,\, \ldots,\, \mathbf{y}+\mathbf{z},\, \ldots,\, \mathbf{x}_q) = G(\mathbf{x}_1,\, \mathbf{x}_2,\, \ldots,\, \mathbf{y},\, \ldots,\, \mathbf{x}_q) + G(\mathbf{x}_1,\, \mathbf{x}_2,\, \ldots,\, \mathbf{z},\, \ldots,\, \mathbf{x}_q)$, on the first coordinate of H, we can rewrite (3.5) as

$$\int_{\triangle_n} H(\mathbf{x},\, \mathbf{x},\, \ldots,\, \mathbf{x})\, dx = n!\, vol(\triangle_n)\times$$
$$\int_{\Omega_n} H(\lambda_1\mathbf{x}_1,\, \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_n\mathbf{x}_n + (1-\sum_{i=1}^{n}\lambda_i)\mathbf{x}_0, \ldots,$$
$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_n\mathbf{x}_n + (1-\sum_{i=1}^{n}\lambda_i)\mathbf{x}_0)d\lambda +$$
$$\int_{\Omega_n} H(\lambda_2\mathbf{x}_2,\, \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_n\mathbf{x}_n + (1-\sum_{i=1}^{n}\lambda_i)\mathbf{x}_0, \ldots,$$
$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_n\mathbf{x}_n + (1-\sum_{i=1}^{n}\lambda_i)\mathbf{x}_0)d\lambda + \cdots +$$
$$\int_{\Omega_n} H(\lambda_n\mathbf{x}_n,\, \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_n\mathbf{x}_n + (1-\sum_{i=1}^{n}\lambda_i)\mathbf{x}_0, \ldots,$$
$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_n\mathbf{x}_n + (1-\sum_{i=1}^{n}\lambda_i)\mathbf{x}_0)d\lambda +$$
$$\int_{\Omega_n} H((1-\sum_{i=1}^{n}\lambda_i)\mathbf{x}_0,\, \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_n\mathbf{x}_n + (1-\sum_{i=1}^{n}\lambda_i)\mathbf{x}_0, \ldots,$$
$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_n\mathbf{x}_n + (1-\sum_{i=1}^{n}\lambda_i)\mathbf{x}_0)d\lambda.$$

The same expansion that was shown in the first term above is now performed on all q entries of the function H. After all of these expansions have been performed we will obtain a sum of separate integrals of the function H, each of which has q entries. Each entry will now be a monomial, instead of a polynomial as a result of the expansion. In each term, each of the q entries will consist of n x values, $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ with their corresponding constant, $\left(1-\sum_{i=1}^{n}\lambda_i\right), \lambda_1, \lambda_2, \ldots, \lambda_n$. Therefore the possible entries are $\mathbf{x}_0\left(1-\sum_{i=1}^{n}\lambda_i\right), \mathbf{x}_1\lambda_1, \mathbf{x}_2\lambda_2, \ldots, \mathbf{x}_n\lambda_n$. In each individual term, the q entries are made up of some linear combination of these $n+1$ possible entries. Note that the order of the entries is not important because this is a symmetric multilinear form. Knowing this, we can deduce an equation for a general term in this sum. Let $a_0, a_1, \ldots, a_n$ be integers such that $0 \le a_0, a_1, \ldots, a_n \le q$ and $a_0 + a_1 + \ldots + a_n = q$, then we can express a generic term of this summation as

$$\int_{\Omega_n} H\left(\left(\left(1-\sum_{i=i}^{n}\lambda_i\right)\mathbf{x}_0\right)^{a_0}, (\lambda_1\mathbf{x}_1)^{a_1}, (\lambda_2\mathbf{x}_2)^{a_2}, \ldots, (\lambda_n\mathbf{x}_n)^{a_n}\right) d\lambda \tag{3.6}$$

where the notation $H\left(\left(\left(1-\sum_{i=i}^{n}\lambda_i\right)\mathbf{x}_0\right)^{a_0}, (\lambda_1\mathbf{x}_1)^{a_1}, (\lambda_2\mathbf{x}_2)^{a_2}, \ldots, (\lambda_n\mathbf{x}_n)^{a_n}\right)$means that $\left(1-\sum_{i=i}^{n}\lambda_i\right)\mathbf{x}_0$ is repeated a_0 times, $(\lambda_1\mathbf{x}_1)$ is repeated a_1 times, $\ldots$, $\lambda_n\mathbf{x}_n$ is repeated a_n times.

Now we will use the property of a multilinear form that $G(\mathbf{x}_{1,} c\mathbf{x}_{2,}, \ldots, \mathbf{x}_q) = c \times G(\mathbf{x}_{1,} \mathbf{x}_2, \ldots, \mathbf{x}_q)$ to further expand the generic term in (3.6). Doing so we obtain,

$$\left[\int_{\Omega_n}\left(1-\sum_{i=i}^{n}\lambda_i\right)^{a_0}\lambda_1^{a_1}\lambda_2^{a_2}\cdots\lambda_n^{a_n}\right] \times H\left(\mathbf{x}_0^{a_0}, \mathbf{x}_1^{a_1}, \ldots, \mathbf{x}_n^{a_n}\right) d\lambda$$

Note that due to the change of variables previously, we are integrating with respect to λ. Since the function H is no longer dependent on λ, we can rewrite the generic term as

$$\int_{\Omega_n}\left(1-\sum_{i=i}^{n}\lambda_i\right)^{a_0}\lambda_1^{a_1}\lambda_2^{a_2}\cdots\lambda_n^{a_n}\,d\lambda \times H\left(\mathbf{x}_0^{a_0}, \mathbf{x}_1^{a_1}, \ldots, \mathbf{x}_n^{a_n}\right). \tag{3.7}$$

Now that we have deduced a general term, we need to know how many times each term with unique values of $a_0, a_1, \ldots, a_n$ will occur. Again, since the function is symmetric the exact order of the entries is not relevant. What matters is how many different combinations of entries is possible for each set of values for $a_0, a_1, \ldots, a_n$. This will determine how many times a term with those set values for $a_0, a_1, \ldots, a_n$ is repeated, which will determine the coefficient in front of each term. First note that the function H has q entries, so there are q possible locations for any value. Beginning with a_0, we know that $\mathbf{x}_0$ occurs a_0 times. Since there are q entries total, there are $\binom{q}{a_0}$ ways to obtain a term with with $\mathbf{x}_0$ repeated a_0 times. Continuing onto a_1, there are now $q-a_0$ remaining entries in the function once we have entered x_0. Therefore, there are $\binom{q-a_0}{a_1}$ ways for x_1 to be repeated a_1 times. The same logic is repeated until we get to the final term $\mathbf{x}_n$ for which there are $\binom{q-\sum_{i=0}^{n-1}}{a_n}$ different combinations. Finally, we multiply these all together combinations together to determine the number of occurrences of each general term. Note that we need to consider all possible combinations of values for $a_0, a_1, \ldots, a_n$ so that we include all terms in the equation. Therefore, we can rewrite (3.7) as

$$\int_{\triangle_n} H(\mathbf{x},\, \mathbf{x},\, \ldots,\, \mathbf{x})\, d\mathbf{x} = n!\, vol(\triangle_n)\times \tag{3.8}$$
$$\sum_{\sum_0^n a_i = q} \binom{q}{a_0}\binom{q-a_0}{a_1}\cdots \binom{q-\sum_{i=0}^{n-1} a_i}{a_n} H\left(\mathbf{x}_0^{a_0},\, \mathbf{x}_1^{a_1},\, \ldots,\, \mathbf{x}_n^{a_n}\right)$$
$$\int_{\Omega_n}\left(1-\sum_{i=i}^{n}\lambda_i\right)^{a_0} \lambda_1^{a_1}\lambda_2^{a_2}\cdots\lambda_n^{a_n}\, d\lambda.$$

To help clarify this expansion, we will consider the two dimensional case in the next section. For now, we will focus on the integral in the second half of the above expression. Since this integral is being taken over the canonical simplex, we can use (3.1) given at the beginning of the proof for integrating over the canonical simplex. Using this equation, we obtain

$$\int_{\Omega_n}\left(1-\sum_{i=i}^{n}\lambda_i\right)^{a_0} \lambda_1^{a_1}\lambda_2^{a_2}\cdots\lambda_n^{a_n}\, d\lambda = \frac{a_0!\cdots a_n!}{(n+\sum_i a_i)!}.$$

Note that when we initially introduced the variables $a_0,\, a_1, \ldots,\, a_n$, we specified that $a_0 + a_1 + \ldots + a_n = q$. Therefore, $\sum_i a_i = q$, and

$$\int_{\Omega_n}\left(1-\sum_{i=i}^{n}\lambda_i\right)^{a_0} \lambda_1^{a_1}\lambda_2^{a_2}\cdots\lambda_n^{a_n}\, d\lambda = \frac{a_0!\cdots a_n!}{(n+q)!}.$$

Now we will substitute this expression back into (3.8) and simplify. Doing so we obtain,

$$\int_{\triangle_n} H(x,\, x,\, \ldots,\, x)\, dx = n!\, vol(\triangle_n)\times \tag{3.9}$$
$$\sum_{\sum_0^n a_i = q} \binom{q}{a_0}\binom{q-a_0}{a_1}\cdots \binom{q-\sum_{i=0}^{n-1} a_i}{a_n} H\left(x_0^{a_0},\, x_1^{a_1},\, \ldots,\, x_n^{a_n}\right)\frac{a_0!\cdots a_n!}{(n+q)!}.$$

Expanding the binomial terms, we obtain

$$\int_{\triangle_n} H(\mathbf{x},\, \mathbf{x},\, \ldots,\, \mathbf{x})\, d\mathbf{x} = n!\, vol(\triangle_n)\times$$
$$\sum_{\sum_0^n a_i = q}\frac{q!}{(q-a_0)!a_0!}\times\frac{(q-a_0)!}{(q-a_0-a_1)!a_1!}\times\cdots\times\frac{\left(q-\sum_{i=0}^{n-1} a_i\right)!}{(q-\sum_{i=0}^{n} a_i)!a_n!}\times H\left(\mathbf{x}_0^{a_0},\, \mathbf{x}_1^{a_1},\, \ldots,\, \mathbf{x}_n^{a_n}\right)\frac{a_0!\cdots a_n!}{(n+q)!}.$$

Again note that $\sum_{i=0}^n a_i = q$, so $(q-\sum_{i=0}^n a_i)! = (q-q)! = 1$. Therefore, when we cross cancel factorials and simplify we get

$$\begin{aligned}\int_{\triangle_n} H(\mathbf{x},\, \mathbf{x},\, \ldots,\, \mathbf{x})\, d\mathbf{x} &= vol(\triangle_n)\times\frac{n!\,q!}{(n+q)!}\times\sum_{\sum_0^n a_i=q} H\left(\mathbf{x}_0^{a_0},\, \mathbf{x}_1^{a_1},\, \ldots,\, \mathbf{x}_n^{a_n}\right).\\ &= \frac{vol(\triangle_n)}{\binom{n+q}{q}}\times\sum_{\sum_0^n a_i=q} H\left(\mathbf{x}_0^{a_0},\, \mathbf{x}_1^{a_1},\, \ldots,\, \mathbf{x}_n^{a_n}\right)\end{aligned}$$

Finally, we can rewrite the final expression and remove $a_0, a_1, \ldots, a_n$. This gives us

$$\int_{\triangle_n} H(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x})\, d\mathbf{x} = \frac{vol(\triangle_n)}{\binom{n+q}{q}} \times \sum_{0 \le i_1 \cdots \le i_q \le n} H\left(\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, \ldots, \mathbf{x}_{i_q}\right).$$

This concludes the proof. □

Related to this theorem is the work of Baldoni et al. in the paper entitled "How to integrate a polynomial over a simplex"[2]. In section 4.1 of this paper, the authors combine the second version of the polarization identity with the Lasserre-Avrachenkov theorem to give a formula that allows for the integration of any homogenous polynomial (not necessarily a multilinear form) over a simplex. This changes the calculation from two separate steps of first polarizing and then using the Lasserre-Avrachenkov equation to only one step. The proof from Baldoni et al. is adapted below.

Theorem. *2 Let f be a homogeneous polynomial of degree q in n variables, and let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ be the vertices of a n-dimensional simplex $\triangle_n$. Then*

$$\int_{\triangle} f(\mathbf{y})\, d\mathbf{y} = \frac{vol(\triangle_n)}{2^q q! \binom{n+q}{q}} \times \sum_{0 \le i_0 \cdots \le i_q \le n} \sum_{\epsilon \in \pm 1^q} \epsilon_1 \epsilon_2 \cdots \epsilon_q f\left(\sum_{k=1}^{q} \epsilon_k \mathbf{x}_{i_k}\right).$$

Proof. We will start with the polarization identity and then use the Lasserre-Avrachenkov equation. We start with the polarization identity given in [2] (17) which states that any polynomial f that is homogeneous of degree q can be written as a symmetric multilinear form H such that

$$H(\mathbf{x}_1, \ldots, \mathbf{x}_q) = \frac{1}{2^q q!} \sum_{\epsilon \in \pm 1^q} \epsilon_1 \epsilon_2 \cdots \epsilon_q\, f\left(\sum_{i=1}^{q} \epsilon_i \mathbf{x}_i\right).$$

This form is equal to f on the diagonal. We now plug this formula into the Lasserre Avrachenkov equation to get,

$$\begin{aligned} \int_{\triangle} f(\mathbf{y})\, d\mathbf{y} &= \frac{vol(\triangle_n)}{\binom{n+q}{q}} \times \sum_{0 \le i_1 \cdots \le i_q \le n} \left[\frac{1}{2^q q!} \sum_{\epsilon \in \pm 1^q} \epsilon_1 \epsilon_2 \cdots \epsilon_q\, f\left(\sum_{k=1}^{q} \epsilon_k \mathbf{x}_{i_k}\right)\right] \\ &= \frac{vol(\triangle_n)}{2^q q! \binom{n+q}{q}} \times \sum_{0 \le i_0 \cdots \le i_q \le n} \sum_{\epsilon \in \pm 1^q} \epsilon_1 \epsilon_2 \cdots \epsilon_q f\left(\sum_{k=1}^{q} \epsilon_k \mathbf{x}_{i_k}\right). \end{aligned}$$

This concludes the proof. □

Note that we can cut the number of terms in this calculation in half by restricting $\epsilon_1 = +1$ and multiplying the final result by 2. This occurs since the equation is symmetric and will yield the same result regardless of whether $\epsilon_1 = -1$ or $\epsilon_1 = 1$. Therefore, we can eliminate half of the terms by restricting $\epsilon_1 = +1$ and multiplying by 2. Examples of this equation can be found in the Appendix where R output is given. Since the Baldoni form does not require that we calculate H directly, the computer program doesn't deal with H, but rather calculates the values directly from f. Also, the computer program sets $\epsilon_1 = 1$ and multiplies the final result by 2 to reduce the number of evaluations.

4. Examples of Integration Methods

Example 1. Expansion using the properties of a symmetric multilinear form when $n = 2$ and $q = 2$. We begin by plugging in $n = 2$ and $q = 2$ into (3.5) which gives

$$\int_{\triangle_n} H(\mathbf{x},\,\mathbf{x})\,d\mathbf{x} = 2!\,vol(\triangle_2)\times\int_{\Omega_n} H\left(\lambda_1\mathbf{x}_1+\lambda_2\mathbf{x}_2+(1-\lambda_1-\lambda_2)\mathbf{x}_0,\,\lambda_1\mathbf{x}_1+\lambda_2\mathbf{x}_2+(1-\lambda_1-\lambda_2)\mathbf{x}_0\right). \tag{4.1}$$

Using the principle of multilinear forms that $G(\mathbf{x}_1,\mathbf{x}_2,\,\dots,\,u{+}v,\,\dots,\,\mathbf{x}_q) = G(\mathbf{x}_1,\,\mathbf{x}_2,\,\dots,\,u,\,\dots,\,\mathbf{x}_q)+$ $G(\mathbf{x}_1,\,\mathbf{x}_2,\,\dots,\,v,\,\dots,\,\mathbf{x}_q)$ we can expand (4.1) to

$$\int_{\triangle_n} H(\mathbf{x},\,\mathbf{x})\,d\mathbf{x} = 2!\,vol(\triangle_2)\times\{\int_{\Omega_n} H\left(\lambda_1\mathbf{x}_1,\,\lambda_1\mathbf{x}_1+\lambda_2\mathbf{x}_2+(1-\lambda_1-\lambda_2)\mathbf{x}_0\right)d\lambda+$$
$$\int_{\Omega_n} H\left(\lambda_2\mathbf{x}_2,\,\lambda_1\mathbf{x}_1+\lambda_2\mathbf{x}_2+(1-\lambda_1-\lambda_2)\mathbf{x}_0\right)d\lambda+$$
$$\int_{\Omega_n} H\left((1-\lambda_1-\lambda_2)\mathbf{x}_0,\,\lambda_1\mathbf{x}_1+\lambda_2\mathbf{x}_2+(1-\lambda_1-\lambda_2)\mathbf{x}_0\right)d\lambda\}.$$

Using the same property on the second entry we obtain

$$\int_{\triangle_n} H(\mathbf{x},\,\mathbf{x})\,d\mathbf{x} = 2!\,vol(\triangle_2)\times\{\int_{\Omega_n} H\left(\lambda_1\mathbf{x}_1,\lambda_1\mathbf{x}_1\right)d\lambda+\int_{\Omega_n} H\left(\lambda_1\mathbf{x}_1,\lambda_2\mathbf{x}_2\right)d\lambda+$$
$$\int_{\Omega_n} H\left(\lambda_1\mathbf{x}_1,(1-\lambda_1-\lambda_2)\mathbf{x}_0\right)d\lambda+\int_{\Omega_n} H\left(\lambda_2\mathbf{x}_2,\lambda_1\mathbf{x}_1\right)d\lambda+\int_{\Omega_n} H\left(\lambda_2\mathbf{x}_2,\lambda_2\mathbf{x}_2\right)d\lambda+$$
$$\int_{\Omega_n} H\left(\lambda_2\mathbf{x}_2,(1-\lambda_1-\lambda_2)\mathbf{x}_0\right)d\lambda+\int_{\Omega_n} H\left((1-\lambda_1-\lambda_2)\mathbf{x}_0,\,\lambda_1\mathbf{x}_1\right)d\lambda+$$
$$\int_{\Omega_n} H\left((1-\lambda_1-\lambda_2)\mathbf{x}_0,\,\lambda_2\mathbf{x}_2\right)d\lambda+\int_{\Omega_n} H\left((1-\lambda_1-\lambda_2)\mathbf{x}_0,\,(1-\lambda_1-\lambda_2)\mathbf{x}_0\right)d\lambda\}.$$

Now we will use the property of multilinear forms that $G(\mathbf{x}_1,c\mathbf{x}_2,,\,\dots,\,\mathbf{x}_q) = c\times$ $G(\mathbf{x}_1,c\mathbf{x},\,\dots,\,\mathbf{x}_q)$ to rewrite this as

$$\int_{\triangle_n} H(\mathbf{x},\,\mathbf{x})\,d\mathbf{x} = 2!\,vol(\triangle_2)\times\{\int_{\Omega_n} \lambda_1\lambda_1 d\lambda H\left(\mathbf{x}_1,\mathbf{x}_1\right)+\lambda_1\lambda_2 d\lambda H\left(\mathbf{x}_1,\mathbf{x}_2\right)+\lambda_1(1-\lambda_1-\lambda_2)d\lambda H\left(\mathbf{x}_1,\,\mathbf{x}_0\right)+$$
$$\lambda_2\lambda_1 d\lambda H\left(\mathbf{x}_2,\mathbf{x}_1\right)+\lambda_2\lambda_2 d\lambda H\left(\mathbf{x}_2,\mathbf{x}_2\right)+\lambda_2(1-\lambda_1-\lambda_2)d\lambda H\left(\mathbf{x}_2,\mathbf{x}_0\right)+$$
$$(1-\lambda_1-\lambda_2)\lambda_1 d\lambda H\left(\mathbf{x}_0,\,\mathbf{x}_1\right)+(1-\lambda_1-\lambda_2)\lambda_2 d\lambda H\left(\mathbf{x}_0,\,\mathbf{x}_2\right)+(1-\lambda_1-\lambda_2)(1-\lambda_1-\lambda_2)d\lambda H\left(\mathbf{x}_0,\,\mathbf{x}_0\right)\}.$$

Since this is a symmetric multilinear form, the order of the entries do not matter, so we can combine like terms to get

$$\int_{\triangle_n} H(\mathbf{x},\,\mathbf{x})\,d\mathbf{x} = 2!\,vol(\triangle_2)\times \tag{4.2}$$
$$\{\int_{\Omega_n} \lambda_1\lambda_1 d\lambda\, H\left(\mathbf{x}_1,\mathbf{x}_1\right)+\int_{\Omega_n} \lambda_2\lambda_2 d\lambda\, H\left(\mathbf{x}_2,\mathbf{x}_2\right)+\int_{\Omega_n} (1-\lambda_1-\lambda_2)(1-\lambda_1-\lambda_2)d\lambda\, H\left(\mathbf{x}_0,\,\mathbf{x}_0\right)$$
$$+\int_{\Omega_n} 2\lambda_1\lambda_2 d\lambda\, H\left(\mathbf{x}_1,\mathbf{x}_2\right)+\int_{\Omega_n} 2\lambda_1(1-\lambda_1-\lambda_2)d\lambda\, H\left(\mathbf{x}_1,\,\mathbf{x}_0\right)+\int_{\Omega_n} 2\lambda_2(1-\lambda_1-\lambda_2)d\lambda\, H\left(\mathbf{x}_2,\mathbf{x}_0\right)\}$$

Note that this is the same as (3.7) for $q = 2$ and $n = 2$. Expression (3.8) gives

$$\int_{\triangle_n} H(\mathbf{x},\, \mathbf{x},\, \ldots,\, \mathbf{x})\, d\mathbf{x} \;= n!\, vol(\triangle_n)\times \tag{4.3}$$
$$\sum_{\sum_0^n a_i = q} \binom{q}{a_0}\binom{q-a_0}{a_1}\cdots\binom{q-\sum_{i=0}^{n-1} a_i}{a_n} H\left(\mathbf{x}_0^{a_0},\, \mathbf{x}_1^{a_1},\, \ldots,\, \mathbf{x}_n^{a_n}\right) \int_{\Omega_n} \left(1-\sum_{i=i}^{n} \lambda_i\right)^{a_0} \lambda_1^{a_1}\lambda_2^{a_2}\cdots\lambda_n^{a_n}\, d\lambda.$$

for $a_0,\, a_1, \ldots,\, a_n \in \mathbb{Z}$ such that $0 \le a_0,\, a_1, \ldots,\, a_n \le q$ and $a_0 + a_1 + \ldots + a_n = q$. When $q = 2$ and $n = 2$, (3.7) becomes

$$\int_{\triangle_n} H(\mathbf{x},\, \mathbf{x})\, d\mathbf{x} \;= 2!\, vol(\triangle_2)\times$$
$$\sum_{\sum_0^2 a_i = 2} \binom{2}{a_0}\binom{2-a_0}{a_1}\binom{2-a_0-a_1}{a_2} \int_{\Omega_n} (1-\lambda_1-\lambda_2)^{a_0}\lambda_1^{a_1}\lambda_2^{a_2}\, d\lambda \times H\left(\mathbf{x}_0^{a_0},\, \mathbf{x}_1^{a_1},\, \mathbf{x}_2^{a_2}\right).$$

Expanding this, we will get the result from (4.2) as we would expect.

Example 2. Integrating on the canonical simplex using the Lasserre-Avrachenkov equation.

The equation we will use to calculate an exact value for the integral is

$$\int_{\Omega_n} x_1^{\alpha_1}\cdots x_n^{\alpha_n}\, dx \;= \frac{\alpha_1!\cdots\, \alpha_n!}{(n+\sum_i \alpha_i)!}.$$

We will be working in three dimensions and calculating the integral of

$$\int_{\Omega_n} x_1^4\, x_2^2\, x_3\, dx$$

Using the equation we can simply plug in the values of the exponents, which gives us

$$\begin{aligned} \int_{\Omega_n} x_1^4\, x_2^2\, x_3\, dx \quad &= \quad \frac{4!\, 2!\, 1!}{(3+4+2+1)!} \\ &= \quad \frac{1}{75600}. \end{aligned}$$

Clearly, this is a very straightforward and simple way for calculating the integral. Furthermore, as the dimensions increase the equation does not become significantly more computationally difficult, making this a very useful equation particularly in higher dimensions. However, the limitation of this equation is that it can only be used on the canonical simplex.

Example 3. Traditional integration on the canonical simplex

We will now calculate the same integral as above using traditional integration methods as follows

$$\begin{aligned}
\int_{\Omega_n} x_1^4\, x_2^2\, x_3\, dx &= \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} x_1^4\, x_2^2\, x_3\, dx_3 dx_2 dx_1 \\
&= \frac{1}{2}\left[\int_0^1 \int_0^{1-x_1} x_1^4\, x_2^2\, (1-x_1-x_2)^2\, dx_2 dx_1\right] \\
&= \frac{1}{2}\left[\int_0^1 \int_0^{1-x_1} x_1^6\, x_2^2 + 2x_1^5 x_2^3 - 2x_1^5 x_2^2 + x_1^4 x_2^4 - 2x_1^4 x_2^3 + x_1^4 x_2^2\, dx_2 dx_1\right] \\
&= \frac{1}{2}[\int_0^1 \frac{x_1^6\,(1-x_1)^3}{3} + \frac{x_1^5\,(1-x_1)^4}{2} - \frac{2x_1^5\,(1-x_1)^3}{3} + \\
&\qquad \frac{x_1^4\,(1-x_1)^5}{5} - \frac{x_1^4\,(1-x_1)^4}{2} + \frac{x_1^4\,(1-x_1)^3}{3}]\, dx_1 \\
&= \frac{1}{2}\left[\int_0^1 \frac{-x^9}{30} + \frac{x^8}{6} - \frac{x^7}{3} + \frac{x^6}{3} - \frac{x^5}{6} + \frac{x^4}{30}\, dx_1\right] \\
&= \frac{1}{2}\left[\frac{-1}{300} + \frac{1}{54} - \frac{1}{24} + \frac{1}{21} - \frac{1}{36} + \frac{1}{150}\right] \\
&= \frac{1}{75600}.
\end{aligned}$$

Even in this slightly abbreviated form, with some steps and expansions omitted, it is clear that this is a much more complicated calculation than that of Example 3. Furthermore, it is clear to see that as the dimensions increase this calculation will quickly become more lengthy and computationally challenging.

Example 4. Using the first polarization formula on a homogenous polynomial in $\mathbb{R}^2$

The polarization formula given in the preliminary section gives that

$$H\left(\mathbf{x}_1,\, \ldots,\, \mathbf{x}_q\right) = \frac{1}{q!}\frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_q} f\left(\lambda_1 \mathbf{x}_1 + \ldots + \lambda_q \mathbf{x}_q\right)|_{\lambda=0}. \tag{4.4}$$

where f is a homogenous polynomial of degree q and H is a symmetric multilinear form. The polynomial we will use in this example is

$$f(x_1,\, x_2) = 4x_1^2 + 5x_1 x_2 + x_2^2.$$

The polynomial is homogenous of order 2, $q = 2$, $n = 2$. Plugging into (4.4) we get

$$H(\mathbf{x}_1,\, \mathbf{x}_2) = \frac{1}{2!}\frac{\partial}{\partial \lambda_1}\frac{\partial}{\partial \lambda_2} f\left(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2\right)|_{\lambda=0}$$

Also, note that the notation we will use to represent the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\mathbf{x}_1 = \begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix} \text{ and } \mathbf{x}_2 = \begin{pmatrix} x_2^{(1)} \\ x_2^{(2)} \end{pmatrix}$$

Plugging this is in we obtain,

$$H(\mathbf{x}_1,\,\mathbf{x}_2)=\frac{1}{2!}\frac{\partial}{\partial\lambda_1}\frac{\partial}{\partial\lambda_2}f\begin{pmatrix}\lambda_1x_1^{(1)}+\lambda_2x_2^{(1)}\\ \lambda_1x_1^{(2)}+\lambda_2x_2^{(2)}\end{pmatrix}_{\lambda=0}$$

$$=\frac{1}{2!}\frac{\partial}{\partial\lambda_1}\frac{\partial}{\partial\lambda_2}\left\{4\left(\lambda_1x_1^{(1)}+\lambda_2x_2^{(1)}\right)^2+5\left(\lambda_1x_1^{(1)}+\lambda_2x_2^{(1)}\right)\left(\lambda_1x_1^{(2)}+\lambda_2x_2^{(2)}\right)+1\left(\lambda_1x_1^{(2)}+\lambda_2x_2^{(2)}\right)^2\right\}_{\lambda=0}$$

Differentiating using the product rule we get

$$=\frac{1}{2!}\frac{\partial}{\partial\lambda_1}\{8\left(\lambda_1x_1^{(1)}+\lambda_2x_2^{(1)}\right)^1\cdot x_2^{(1)}+$$

$$5\left\{\left(x_2^{(1)}\right)\left(\lambda_1x_1^{(2)}+\lambda_2x_2^{(2)}\right)+\left(x_2^{(2)}\right)\left(\lambda_1x_1^{(1)}+\lambda_2x_2^{(1)}\right)\right\}+2\left(\lambda_1x_1^{(2)}+\lambda_2x_2^{(2)}\right)^2\left(x_2^{(2)}\right)\}_{\lambda=0}$$

$$=\frac{1}{2}\left[8x_2^{(1)}x_1^{(1)}+5\left(x_2^{(1)}x_1^{(2)}+x_1^{(1)}x_2^{(2)}\right)+2x_1^{(2)}x_2^{(2)}\right]_{\lambda=0}$$

Note that there are no longer any λ terms remaining. Therefore, the final equation is

$$H(\mathbf{x}_1,\,\mathbf{x}_2)=4x_2^{(1)}x_1^{(1)}+\frac{5}{2}\left(x_2^{(1)}x_1^{(2)}+x_1^{(1)}x_2^{(2)}\right)+x_1^{(2)}x_2^{(2)}.$$

Example 5. Using the second polarization formula on a homogenous polynomial in $\mathbb{R}^3$

The second polarization formula given in the preliminary section states that

$$H\left(\mathbf{x}_1,\,\ldots,\,\mathbf{x}_q\right)=\frac{1}{2^qq!}\sum_{\epsilon\in\pm1^q}\epsilon_1\epsilon_2\cdots\epsilon_q\,f\left(\sum_{i=1}^{q}\epsilon_i\mathbf{x}_i\right).$$

We will now use this polarization formula to determine the symmetric multilinear form of the homogenous polynomial

$$f(x_1,\,x_2,\,x_3)=3x_1^2+4x_2x_3.$$

Since the polynomial is homogenous of degree 2, $q=2$, $n=3$. Also note that the notation used for the vectors $\mathbf{x}_1$and $\mathbf{x}_2$ is

$$\mathbf{x}_1=\begin{pmatrix}x_1^{(1)}\\ x_1^{(2)}\\ x_1^{(3)}\end{pmatrix}\text{ and }\mathbf{x}_2=\begin{pmatrix}x_2^{(1)}\\ x_2^{(2)}\\ x_2^{(3)}\end{pmatrix}$$

Plugging these in we obtain,

$$\begin{aligned} H(\mathbf{x}_1,\, \mathbf{x}_2) &= \frac{1}{2^2 2!} \sum_{\epsilon\in\{\pm1\}^2} \epsilon_1\epsilon_2\, f\left(\epsilon_1 x_1^{(1)} + \epsilon_2 x_2^{(1)},\, \epsilon_1 x_1^{(2)} + \epsilon_2 x_2^{(2)},\, \epsilon_1 x_1^{(3)} + \epsilon_2 x_2^{(3)}\right) \\ &= \frac{1}{8} \sum_{\epsilon\in\{-1,1\}} \epsilon_1\epsilon_2 \{3\left(\epsilon_1^2 x_1^{(1)^2} + 2\epsilon_1\epsilon_2 x_1^{(1)} x_2^{(1)} + \epsilon_2^2 x_2^{(1)^2}\right) + \\ &\quad 4\left(\epsilon_1^2 x_1^{(2)} x_1^{(3)} + \epsilon_1\epsilon_2 x_1^{(2)} x_2^{(3)} + \epsilon_1\epsilon_2 x_2^{(2)} x_1^{(3)} + \epsilon_2^2 x_2^{(3)^2}\right)\} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{8}\{(1)(1)\left(3x_1^{(1)^2} + 6x_1^{(1)}x_2^{(1)} + 3x_2^{(1)^2} + 4x_1^{(2)}x_1^{(3)} + 4x_1^{(2)}x_2^{(3)} + 4x_2^{(2)}x_1^{(3)} + 4x_2^{(3)^2}\right) + \\ &(-1)(-1)\left(3x_1^{(1)^2} + 6x_1^{(1)}x_2^{(1)} + 3x_2^{(1)^2} + 4x_1^{(2)}x_1^{(3)} + 4x_1^{(2)}x_2^{(3)} + 4x_2^{(2)}x_1^{(3)} + 4x_2^{(3)^2}\right) + \\ &(1)(-1)\left(3x_1^{(1)^2} - 6x_1^{(1)}x_2^{(1)} + 3x_2^{(1)^2} + 4x_1^{(2)}x_1^{(3)} - 4x_1^{(2)}x_2^{(3)} - 4x_2^{(2)}x_1^{(3)} + 4x_2^{(3)^2}\right) + \\ &(-1)(1)\left(3x_1^{(1)^2} - 6x_1^{(1)}x_2^{(1)} + 3x_2^{(1)^2} + 4x_1^{(2)}x_1^{(3)} - 4x_1^{(2)}x_2^{(3)} - 4x_2^{(2)}x_1^{(3)} + 4x_2^{(3)^2}\right)\} \\ &= 3x_1^{(1)}x_2^{(1)} + 2x_1^{(2)}x_2^{(3)} + 2x_2^{(2)}x_1^{(3)}. \end{aligned}$$

Thus, the symmetric multilinear form of $f(x_1,\, x_2,\, x_3) = 3x_1^2 + 4x_2x_3$ is $H(\mathbf{x}_1,\, \mathbf{x}_2) = 3x_1^{(1)}x_2^{(1)} + 2x_1^{(2)}x_2^{(3)} + 2x_2^{(2)}x_1^{(3)}$.

Example 6. Integrating a homogenous polynomial on a nonstandard simplex using the Lasserre-Avrachenkov equation

We will now use the formula we previously proved to solve an example. The polynomial over which we will take the integral is

$$f(x_1,\, x_2) = 4x_1^2 + 5x_1x_2 + x_2^2.$$

The first step is to re-write the polynomial f as symmetric multilinear form using one of the polarization theorems. However, we have already done this in Example 4 . Therefore, the symmetric multilinear form that we will be using is

$$H(\mathbf{x}_1,\, \mathbf{x}_2) = 4x_2^{(1)}x_1^{(1)} + \frac{5}{2}\left(x_2^{(1)}x_1^{(2)} + x_1^{(1)}x_2^{(2)}\right) + x_1^{(2)}x_2^{(2)}.$$

The simplex, $\triangle$ we will be using has the vertices $\mathbf{x}_0 = (1,\, 2)$, $\mathbf{x}_1 = (2,\, 1)$, and $\mathbf{x}_2 = (3,\, 6)$.

We will begin by determining the volume of the simplex. We can do this by finding the determinant of the matrix formed by the vertices and a column of ones. Thus,

$$\begin{aligned} vol(\triangle_n) &= \frac{1}{n!}\det(x_1 - x_0,\, x_2 - x_0) \\ &= \frac{1}{2}\det\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \\ &= 3 \end{aligned}$$

Also, note that we are working in two dimensions so $n = 2$, and the original polynomial was homogenous of order 2 so $q = 2$. Plugging this into the integration formula, we obtain

$$\int_{\triangle_n} H(\mathbf{x},\, \mathbf{x})\, d\mathbf{x} = \frac{3}{\binom{4}{2}}\left[\sum_{0\le i_1\le i_2\le 2} H(\mathbf{x}_{i_1},\, \mathbf{x}_{i_2})\right].$$

Expanding the summation, we get

$$\int_{\triangle_n} H(\mathbf{x},\,\mathbf{x})\,d\mathbf{x} \;=\; \frac{3}{6}[H(\mathbf{x}_0,\,\mathbf{x}_0)+H(\mathbf{x}_0,\,\mathbf{x}_1)+H(\mathbf{x}_0,\,\mathbf{x}_2)+$$
$$H(\mathbf{x}_1,\,\mathbf{x}_1)+H(\mathbf{x}_1,\,\mathbf{x}_2)+H(\mathbf{x}_2,\,\mathbf{x}_2)].$$

Plugging in $\mathbf{x}_0 = \begin{pmatrix}1\\2\end{pmatrix}$, $\mathbf{x}_1 = \begin{pmatrix}2\\1\end{pmatrix}$, and $\mathbf{x}_2 = \begin{pmatrix}3\\6\end{pmatrix}$ we get

$$\int_{\triangle_n} H(\mathbf{x},\,\mathbf{x})\,d\mathbf{x} = \frac{1}{2}\{\left[4(1)(1)+\frac{5}{2}\left((1)(2)+(1)(2)\right)+(2)(2)\right]+\left[4(2)(1)+\frac{5}{2}\left((2)(2)+(1)(1)\right)+(1)(2)\right]+$$
$$\left[4(3)(1)+\frac{5}{2}\left((3)(2)+(1)(6)\right)+(2)(6)\right]+\left[4(2)(2)+\frac{5}{2}\left((2)(1)+(2)(1)\right)+(1)(1)\right]+$$
$$\left[4(3)(2)+\frac{5}{2}\left((3)(1)+(2(6)\right)+(1)(6)\right]+\left[4(3)(3)+\frac{5}{2}\left((3)(6)+(3)(6)\right)+(6)(6)\right]\}$$

$$\begin{aligned}\int_{\triangle_n} H(\mathbf{x},\,\mathbf{x})\,d\mathbf{x} &= \frac{1}{2}\left(18+\frac{45}{2}+54+27+\frac{135}{2}+162\right)\\ &= \frac{351}{2}\end{aligned}$$

Although at first glance this might look somewhat complex, it is important to note that each of the steps performed here is actually quite simple. The calculations are based on evaluating the polynomial at specified points and determining the constant up front. In addition, the simplex did not need to be parametrized and split into easily expressed sub-regions, a task that becomes increasingly difficult in higher dimensions. Consequently, all of the steps performed here can be done on a computer, so it is possible to write a computer program to evaluate these types of integrals, which greatly increases the speed of calculation.

Example 7. Integrating a homogenous polynomial on a nonstandard simplex using the traditional method.

We will now determine the value of the same integral we calculated in the previous example, but this time by traditional integration methods. As a reminder, the simplex over which we will be integrating has the vertices (1, 2), (2, 1), and (3, 6), and the polynomial we will be using is $f(x_1,\,x_2) = 4x_1^2+5x_1x_2+x_2^2$. The first step in performing this integral by traditional methods is to determine the bounds of integration by parametrization. Although this may not seem too difficult in this example, since we are only working in two dimensions, keep in mind that this step will become increasingly more complicated at the dimensions increase and the simplex become impossible to visualize. Using calculus we can calculate,

$$\iint_{\triangle} 4x_1^2 + 5x_1x_2 + x_2^2 dx_1 dx_2$$

$$= \int_1^2 \int_{-x_1+3}^{2x_1} 4x_1^2 + 5x_1x_2 + x_2^2 dx_2 dx_1 + \int_2^3 \int_{5x_1-9}^{2x_1} 4x_1^2 + 5x_1x_2 + x_2^2 dx_2 dx_1$$

$$= \int_1^2 \left(4x_1^2(2x_1) + \frac{5}{2}x_1(2x_1)^2 + \frac{1}{3}(2x_1)^3\right) - \left(4x_1^2(-x_1+3) + \frac{5}{2}x_1(-x_1+3)^2 + \frac{1}{3}(-x_1+3)^3\right) dx_1 +$$

$$\int_2^3 \left(4x_1^2(2x_1) + \frac{5}{2}x_1(2x_1)^2 + \frac{1}{3}(2x_1)^3\right) - \left(4x_1^2(5x_1-9) + \frac{5}{2}x_1(5x_1-9)^2 + \frac{1}{3}(5x_1-9)^3\right) dx_1$$

$$= \int_1^2 \left(\frac{45}{2}x_1^3 - \frac{27}{2}x_1 - 9\right) + \int_2^3 \left(-\frac{207}{2}x_1^3 + 486x_1^2 - \frac{1215}{2}x_1 + 243\right) dx_1$$

$$= \left\{\left(\frac{45(2)^4}{8} - \frac{27(2)^2}{4} - 9(2)\right) - \left(\frac{45(1)^4}{8} - \frac{27(1)^2}{4} - 9(1)\right)\right\} +$$

$$\left\{\left(\frac{-207(3)^4}{8} + \frac{486(3)^3}{3} - \frac{1215(3)^2}{4} + 243(3)\right) - \left(\frac{-207(2)^4}{8} + \frac{486(2)^3}{3} - \frac{1215(2)^2}{4} + 243(2)\right)\right\}$$

$$= \frac{351}{2}$$

In conclusion, although this may seem relatively simple in two dimensions performing this calculation in higher dimensions becomes significantly more challenging, so the Lasserre-Avrachenkov equation is often easier.

Example 8. Integrating a non-homogenous equation on a nonstandard simplex using the Lasserre-Avrachenkov equation

We will now use the Lasserre-Avrachenkov equation to integrate a non-homogenous equation, rather than a homogenous equation. To do this, we will need to decompose the general polynomial into the sum of its homogenous terms. To do so we will simply group together all of the terms whose exponents sum to the same value. For example, we will be working with the polynomial

$$f(x_1, x_2) = 3x_1x_2^2 + 4x_1^2 + 5x_1x_2 + x_2^2 - 2x_1 + 3x_2.$$

We can break this up into three homogenous polynomials namely

$$\begin{aligned} f^1(x_1, x_2) &= 3x_1x_2^2 && (q=3) \\ f^2(x_1, x_2) &= 4x_1^2 + 5x_1x_2 + x_2^2 && (q=2) \\ f^3(x_1, x_2) &= -2x_1 + 3x_2 && (q=1). \end{aligned}$$

We now will apply the Lasserre-Avrachenkov equation to each of these homogenous polynomials separately. The simplex over which we will be integrating is the same one used previously in Examples 6 and 7, with the vertices $\mathbf{x}_0 = (1, 2)$, $\mathbf{x}_1 = (2, 1)$, and $\mathbf{x}_2 = (3, 6)$.

We will begin with $f^1(x_1, x_2) = 3x_1x_2^2$. Using either of the polarization formulas, we can find that the symmetric multilinear form of f^1 is

$$H^1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = x_1^{(1)}x_2^{(2)}x_3^{(2)} + x_2^{(1)}x_1^{(2)}x_3^{(2)} + x_1^{(2)}x_2^{(2)}x_3^{(1)}.$$

Since the original polynomial was homogenous of order three in two dimensions, $q = 3$ and $n = 2$. We also determined in Example 6 that the volume of this simplex is 3. We can plug this information into the Lasserre-Avrachenkov equation to obtain

$$\int_{\triangle_n} H^1(\mathbf{x},\,\mathbf{x},\,\mathbf{x})\,d\mathbf{x} \;=\; \frac{3}{\binom{5}{3}}\left[\sum_{0\le i_1\le i_2\le i_3\le 2} H(\mathbf{x}_{i_1},\,\mathbf{x}_{i_2},\,\mathbf{x}_{i_3})\right].$$

Expanding this we get

$$\begin{aligned}
\int_{\triangle_n} H^1(\mathbf{x},\,\mathbf{x},\,\mathbf{x})\,d\mathbf{x} \;&=\; \frac{3}{10}\{H(\mathbf{x}_0,\,\mathbf{x}_0,\,\mathbf{x}_0)+H(\mathbf{x}_0,\,\mathbf{x}_0,\,\mathbf{x}_1)+H(\mathbf{x}_0,\,\mathbf{x}_0,\,\mathbf{x}_2)+\\
&H(\mathbf{x}_0,\,\mathbf{x}_1,\,\mathbf{x}_1)+H(\mathbf{x}_0,\,\mathbf{x}_1,\,\mathbf{x}_2)+H(\mathbf{x}_0,\,\mathbf{x}_2,\,\mathbf{x}_2)+H(\mathbf{x}_1,\,\mathbf{x}_1,\,\mathbf{x}_1)+\\
&H(\mathbf{x}_1,\,\mathbf{x}_1,\,\mathbf{x}_2)+H(\mathbf{x}_1,\,\mathbf{x}_2,\,\mathbf{x}_2)+H(\mathbf{x}_2,\,\mathbf{x}_2,\,\mathbf{x}_2).\\
&= \frac{3}{10}\{[(1)(2)(2)+(1)(2)(2)+(2)(2)(1)]+[(1)(2)(1)+(1)(2)(1)+(2)(2)(2)]+\\
&[(1)(2)(6)+(1)(2)(6)+(2)(2)(3)]+[(1)(1)(1)+(2)(2)(1)+(1)(2)(2)]+\\
&[(1)(1)(6)+(2)(2)(6)+(2)(1)(3)]+[(1)(6)(6)+(3)(2)(6)+(2)(6)(3)]+\\
&[(2)(1)(1)+(2)(1)(1)+(1)(1)(2)]+[(2)(1)(6)+(1)(2)(6)+(1)(1)(3)]+\\
&[(2)(6)(6)+(3)(1)(6)+(1)(6)(3)]+[(3)(6)(6)+(6)(3)(6)+(6)(6)(3)]\\
&= \frac{1017}{5}
\end{aligned}$$

The next step would normally be to apply the Lasserre-Avrachenkov equation to f^2. However, note that we already did this in Example 6. Therefore, the solution to this step of the calculation is $\frac{351}{2}$. Now we continue to f^3. Since f^3 is a homogenous polynomial of order 1 in two dimensions, $n = 2$ and $q = 1$. Also, using one of the polarization identities, we can rewrite f^3 as

$$H^3(\mathbf{x}) = -2x_1^{(1)} + 3x_1^{(2)}$$

Plugging this into the Lasserre-Avrachenkov equation, we get

$$\begin{aligned}
\int_{\triangle_n} H^3(\mathbf{x})\,d\mathbf{x} \quad&=\quad \frac{3}{\binom{3}{1}}\left[\sum_{0\le i_1\le 2} H(\mathbf{x}_{i_1})\right].\\
&=\quad 1\{H(\mathbf{x}_0)+H(\mathbf{x}_1)+H(\mathbf{x}_2)\}\\
&=\quad [(-2)(1)+(3)(2)]+[(-2)(2)+(3)(1)]+[(-2)(3)+(3)(6)]\\
&=\quad 15
\end{aligned}$$

Since the starting polynomial f is just a sum of the three homogenous polynomials f^1, f^2, and f^3, we simply sum the solutions we got from calculating the values of the integrals separately. Thus

$$\begin{aligned}
\int_{\triangle} f(x_1,\,x_2) \quad&=\quad \int_{\triangle} f^1(x_1,\,x_2)+\int_{\triangle} f^2(x_1,\,x_2)+\int_{\triangle} f^3(x_1,\,x_2)\\
&=\quad \frac{1017}{5}+\frac{351}{2}+15\\
&=\quad \frac{3939}{10}
\end{aligned}$$

Example 9. Integrating a non-homogenous equation over a nonstandard simplex using the traditional method.

We will now perform the exact same calculation as performed in Example 8, except this time we will use traditional integration methods. Specifically, we will be integrating the function $f(x_1, x_2) = 3x_1x_2^2 + 4x_1^2 + 5x_1x_2 + x_2^2 - 2x_1 + 3x_2$ over a simplex with vertices $\mathbf{x}_0 = (1, 2)$, $\mathbf{x}_1 = (2, 1)$, and $\mathbf{x}_2 = (3, 6)$. Using calculus, we get

$$\int\int_{\triangle} 3x_1x_2^2 + 4x_1^2 + 5x_1x_2 + x_2^2 - 2x_1 + 3x_2$$

$$= \int_1^2 \int_{-x_1+3}^{2x} 3x_1x_2^2 + 4x_1^2 + 5x_1x_2 + x_2^2 - 2x_1 + 3x_2 dx_2 dx_1 +$$

$$\int_2^3 \int_{5x_1-9}^{2x} 3x_1x_2^2 + 4x_1^2 + 5x_1x_2 + x_2^2 - 2x_1 + 3x_2 dx_2 dx_1$$

$$= \int_1^2 \left[x_1(2x_1)^3 + 4x_1^2(2x_1) + \frac{5}{2}x_1(2x_1)^2 + \frac{1}{3}(2x)^3 - 2x_1(2x_1) + \frac{3}{2}(2x_1)^2\right] -$$

$$\left[x_1(-x_1+3)^3 + 4x_1^2(-x_1+3) + \frac{5}{2}x_1(-x_1+3)^2 + \frac{1}{3}(-x_1+3)^3 - 2x_1(-x_1+3) + \frac{3}{2}(-x_1+3)^2\right] dx_1 +$$

$$\int_2^3 \left[x_1(2x_1)^3 + 4x_1^2(2x_1) + \frac{5}{2}x_1(2x_1)^2 + \frac{1}{3}(2x)^3 - 2x_1(2x_1) + \frac{3}{2}(2x_1)^2\right] -$$

$$\left[x_1(5x_1-9)^3 + 4x_1^2(5x_1-9) + \frac{5}{2}x_1(5x_1-9)^2 + \frac{1}{3}(5x-9)^3 - 2x_1(5x_1-9) + \frac{3}{2}(5x_1-9)^2\right] dx_1$$

$$= \int_1^2 9x_1^4 + \frac{27}{2}x_1^3 + \frac{51}{2}x_1^2 - \frac{51}{2}x_1 + \frac{45}{2}dx_1 + \int_2^3 -117x_1^4 + \frac{1143}{2}x_1^3 - \frac{1509}{2}x_1^2 + \frac{477}{2}x_1 + \frac{243}{2}dx_1$$

$$= \left[\frac{9(2)^5}{5} + \frac{27(2)^4}{8} + \frac{51(2)^3}{6} - \frac{51(2)^2}{4} - \frac{45}{2}(2)\right] - \left[\frac{9(1)^5}{5} + \frac{27(1)^4}{8} + \frac{51(1)^3}{6} - \frac{51(1)^2}{4} - \frac{45(1)}{2}\right] +$$

$$\left[\frac{-117(3)^5}{5} + \frac{1143(3)^4}{8} - \frac{1509(3)^3}{6} + \frac{477(3)^2}{4} + \frac{243(3)^2}{4} + \frac{243(3)}{2}\right] -$$

$$\left[\frac{-117(2)^5}{5} + \frac{1143(2)^4}{8} - \frac{1509(2)^3}{6} + \frac{477(2)^2}{4} + \frac{243(2)}{2}\right]$$

$$= \frac{3939}{10}.$$

Although we did not need to break up the f into homogenous polynomials using this method, it is important to note that we still had the parametrize the simplex and determine the bounds. It is this step that will become progressively more challenging in higher dimensions, making the Lasserre-Avrachenkov equation the better option.

5. Discussion

Clearly, the Lasserre-Avrachenkov equation is an extremely useful equation for integrating over simplices. By reducing the integration into more computationally simple steps based on the vertices of the simplex, it can greatly speed up the evaluation of these integrals, specifically in higher dimensions. This will allow these integrals to be performed much more efficiently, allowing their results to be used in further research. The Lasserre-Avrachenkov paper has also led to other publications and research into this topic.

Most notably related to the Lasserre-Avrachenkov paper is Khosravifard, Ezmaeili, and Saidi's "Extension of the Lasserre-Avrachenkov theorem on the integral of multilinear forms over simplices" [3]. In this paper, Khosravifard et al. expand on the Lasserre-Avrachenkov equation, proving that it can be used on quasilinear forms in addition to just multilinear forms. Definition 1 from [3] state that a form $\mathbf{Q} : (\mathbb{R}^n)^q \to \mathbb{R}$ is called q-quasilinear if for all $0 \le c \le 1$ and $1 \le i \le q$ we have

$$\begin{aligned}\mathbf{Q}(X_1, \ldots, X_{i-1}, cX_i' + (1-c)X_i", X_{i+1}, \ldots X_q) = c\mathbf{Q}(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots X_q) + \\ (1-c)\mathbf{Q}(X_1, \ldots, X_{i-1}, X_i", X_{i+1}, \ldots X_q).\end{aligned}$$

Since this condition is a more general condition then the requirements of a multilinear form, this allows the Lasserre-Avrachenkov equation to be used with more polynomials. In addition, Khosravifrad et al. discuses a different approach for integration over non-homogenous polynomials that can be simpler and easier in certain cases then decomposing the polynomial.

One key benefit of the Lasserre-Avrachenkov equation is that the calculations are basic enough to be performed by a computer, even in higher dimensions. This benefit has been utilized to develop software to perform the calculation of these integrals automatically. One notable example of this software is described in De Loera et. al's paper entitled "Software for Exact Integration of Polynomials over Polyhedra." Although not strictly limited to simplices, this paper describes a software that utilizes equations similar to the Lasserre-Avrachenkov and Baldoni et. al equations to efficiently calculate the exact values of integrals over higher dimensional shapes. More specifically related to Lasserre-Avrachenkov, Dr. John Nolan has used the R program to create a package that directly uses the Lasserre-Avrachenkov equation to quickly calculate the value of integrals over simplices. Some output of this program, corresponding with the work shown in Example 6 can be found in the Appendix. Particularly when working in higher dimensions, it is useful to note the number of calculations that must be carried out by the computer in order to understand how long it will take. For a homogenous polynomial there are $\frac{2^q}{2}\binom{q+n}{q} = 2^{q-1}\binom{q+n}{q}$ evaluations where q is the degree of homogeneity and n is the dimension.

Finally, although this paper clarifies a lot of the questions that may come up regarding the Lasserre-Avrachenkov equation, there are still opportunities for further research on this subject. Specifically, some further questions that arise are the constraints on q. Must it be a positive integer, or is there a reasonable interpretation of this problem where q is negative or not an integer? In addition, does the same or a similar equation work when a lower dimensional simplex exists in a higher dimensional space. In other words, how are the calculations different when the $dim(\triangle) < n$?

References

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Appendix

R output for the evaluation of $f(x_1, x_2) = 4x_1^2 + 5x_1x_2 + x_2^2$ over a simplex with vertices $\mathbf{x}_1 = (1, 2)$, $\mathbf{x}_2 = (2, 1)$, and $\mathbf{x}_3 = (3, 6)$. Note that this is the same calculation performed in Examples 6 and 7; however, this program uses the Baldoni et al. [2]equation which does not require H to be calculated directly.

```
> f <- definePoly( c(4,5,1), matrix( c(2,0,  1,1,  0,2),
        byrow=TRUE, nrow=3, ncol=2))
> printPoly(f) Polynomial has 3 terms and 2 variables 4 * x[1]^2
        (degree= 2 )   + 5 * x[1]^1 * x[2]^1 (degree= 2 )+ 1 * x[2]^2
        (degree= 2 )
> S <- matrix( c(1,2,  2,1,  3,6), nrow=2, ncol=3)
> integrateSimplexPolynomial( f, S, method="LA" )

f(x)= 72    x= 2 4
f(x)= 0     x= 0 0
f(x)= 90     x= 3 3
f(x)= 0     x= -1 1
f(x)= 288     x= 4 8
f(x)= 72     x= -2 -4
f(x)= 108     x= 4 2
f(x)= 0     x= 0 0
f(x)= 324     x= 5 7
f(x)= 54     x= -1 -5
f(x)= 648     x= 6 12
f(x)= 0     x= 0 0

$integral [1] 175.5
$functionEvaluations [1] 12
```

R output for the evaluation of $f(x_1, x_2) = 4\mathbf{x}_1\mathbf{x}_2^3 + 2\mathbf{x}_1^2\mathbf{x}_2$ over the same simplex.

```
> p <- definePoly( c(4,2,1), matrix( c(1,3, 2,1, 4,0), byrow=TRUE, nrow=3))
> printPoly(p) Polynomial has 3 terms and 2 variables
        4 * x[1]^1 * x[2]^3(degree= 4 )+ 2 * x[1]^2 * x[2]^1
        (degree= 3 )+ 1 * x[1]^4 (degree= 4 )
> integrateSimplexPolynomial( p, S, method="LA" )
f(x)= 8448     x= 4 8
f(x)= 528     x= 2 4
f(x)= 528     x= 2 4
f(x)= 0     x= 0 0
f(x)= 528     x= 2 4
f(x)= 0     x= 0 0
```

```
f(x)= 0      x= 0 0
f(x)= 528      x= -2 -4
f(x)= 7485      x= 5 7
f(x)= 501      x= 1 5
f(x)= 405      x= 3 3
f(x)= -3      x= -1 1
f(x)= 405      x= 3 3
f(x)= -3      x= -1 1
f(x)= -3      x= 1 -1
f(x)= 405      x= -3 -3
f(x)= 42768      x= 6 12
f(x)= 0      x= 0 0
f(x)= 8448      x= 4 8
f(x)= 528      x= -2 -4
f(x)= 8448      x= 4 8
f(x)= 528      x= -2 -4
f(x)= 528      x= 2 4
f(x)= 8448      x= -4 -8
f(x)= 6480      x= 6 6
f(x)= 528      x= 2 4
f(x)= 528      x= 2 4
f(x)= -48      x= -2 2
f(x)= 384      x= 4 2
f(x)= 0      x= 0 0
f(x)= 0      x= 0 0
f(x)= 384      x= -4 -2
f(x)= 39669      x= 7 11
f(x)= -3      x= 1 -1
f(x)= 8829      x= 3 9
f(x)= 405      x= -3 -3
f(x)= 7485      x= 5 7
f(x)= 501      x= -1 -5
f(x)= 501      x= 1 5
f(x)= 7485      x= -5 -7
f(x)= 135168      x= 8 16
f(x)= 528      x= 2 4
f(x)= 528      x= 2 4
f(x)= 8448      x= -4 -8
f(x)= 42768      x= 6 12
f(x)= 0      x= 0 0
f(x)= 0      x= 0 0
f(x)= 42768      x= -6 -12
f(x)= 5901      x= 7 5
f(x)= 405      x= 3 3
f(x)= 405      x= 3 3
```

```
f(x)= -3      x= -1 1
f(x)= 405      x= 3 3
f(x)= -3      x= -1 1
f(x)= -3      x= -1 1
f(x)= 645      x= -5 -1
f(x)= 36096      x= 8 10
f(x)= -48      x= 2 -2
f(x)= 8448      x= 4 8
f(x)= 528      x= -2 -4
f(x)= 8448      x= 4 8
f(x)= 528      x= -2 -4
f(x)= 0      x= 0 6
f(x)= 6480      x= -6 -6
f(x)= 128061      x= 9 15
f(x)= 405      x= 3 3
f(x)= 405      x= 3 3
f(x)= 8829      x= -3 -9
f(x)= 44565      x= 5 13
f(x)= -3      x= -1 1
f(x)= -3      x= -1 1
f(x)= 39669      x= -7 -11
f(x)= 330000      x= 10 20
f(x)= 8448      x= 4 8
f(x)= 8448      x= 4 8
f(x)= 528      x= -2 -4
f(x)= 8448      x= 4 8
f(x)= 528      x= -2 -4
f(x)= 528      x= -2 -4
f(x)= 135168      x= -8 -16
f(x)= 6144      x= 8 4
f(x)= 384      x= 4 2
f(x)= 384      x= 4 2
f(x)= 0      x= 0 0
f(x)= 384      x= 4 2
f(x)= 0      x= 0 0
f(x)= 0      x= 0 0
f(x)= 384      x= -4 -2
f(x)= 32805      x= 9 9
f(x)= -243      x= 3 -3
f(x)= 7485      x= 5 7
f(x)= 501      x= -1 -5
f(x)= 7485      x= 5 7
f(x)= 501      x= -1 -5
f(x)= 501      x= 1 5
f(x)= 7485      x= -5 -7
```

```
f(x)= 119760    x= 10 14
f(x)= 384    x= 4 2
f(x)= 384    x= 4 2
f(x)= 8016    x= -2 -10
f(x)= 42768    x= 6 12
f(x)= 0    x= 0 0
f(x)= 0    x= 0 0
f(x)= 42768    x= -6 -12
f(x)= 316437    x= 11 19
f(x)= 7485    x= 5 7
f(x)= 7485    x= 5 7
f(x)= 501    x= -1 -5
f(x)= 7485    x= 5 7
f(x)= 501    x= -1 -5
f(x)= 501    x= -1 -5
f(x)= 139965    x= -7 -17
f(x)= 684288    x= 12 24
f(x)= 42768    x= 6 12
f(x)= 42768    x= 6 12
f(x)= 0    x= 0 0
f(x)= 42768    x= 6 12
f(x)= 0    x= 0 0
f(x)= 0    x= 0 0
f(x)= 42768    x= -6 -12
f(x)= 108    x= 3 6
f(x)= 4    x= 1 2
f(x)= 4    x= 1 2
f(x)= -4    x= -1 -2
f(x)= 160    x= 4 5
f(x)= 0    x= 0 3
f(x)= 8    x= 2 1
f(x)= -8    x= -2 -1
f(x)= 500    x= 5 10
f(x)= -4    x= -1 -2
f(x)= 108    x= 3 6
f(x)= -108    x= -3 -6
f(x)= 200    x= 5 4
f(x)= 4    x= 1 2
f(x)= 4    x= 1 2
f(x)= 0    x= -3 0
f(x)= 648    x= 6 9
f(x)= 0    x= 0 -3
f(x)= 56    x= 2 7
f(x)= -160    x= -4 -5
f(x)= 1372    x= 7 14
```

```
f(x)= 4      x= 1 2
f(x)= 4      x= 1 2
f(x)= -500      x= -5 -10
f(x)= 216      x= 6 3
f(x)= 8      x= 2 1
f(x)= 8      x= 2 1
f(x)= -8      x= -2 -1
f(x)= 784      x= 7 8
f(x)= -8      x= 1 -4
f(x)= 108      x= 3 6
f(x)= -108      x= -3 -6
f(x)= 1664      x= 8 13
f(x)= 8      x= 2 1
f(x)= 8      x= 2 1
f(x)= -352      x= -4 -11
f(x)= 2916      x= 9 18
f(x)= 108      x= 3 6
f(x)= 108      x= 3 6
f(x)= -108      x= -3 -6

$integral [1] 1206
$functionEvaluations [1] 160
```