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Coral Games and the Core of Cores

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Abstract

Many economically and politically important groups are themselves comprised of groups. Examples of such multi-level group structures include coalition governments, labor confederations and research consortia. In this paper, a model of multi-level group structures and an equilibrium concept are developed. The results establish that individuals and groups will make tradeoffs across levels of the overall structure. Therefore, overall stability and overall instability can both arise from any combination of stable and unstable levels. The framework and results have applications in many fields, including political economy, industrial organization, and environmental economics.
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1 Introduction

Many economically and politically important groups are themselves comprised of groups. Examples of such multi-level group structures include coalition governments, labor confederations and research consortia. Traditional concepts of group stability (e.g. the core, the bargaining set, and the α-core) are not equipped to deal with all levels of a multi-level structure simultaneously. Instead, researchers are forced to assume away the effect of all other levels when analyzing the stability of a given level. The model presented here (called coral games) considers stability of all levels simultaneously. It provides insight into the way groups and individuals make tradeoffs across the levels of a group structure and sheds light on a more complicated concept of stability.

The simplest case of a multi-level group structure involves two levels. On one level individuals play a group formation game. On another level those groups are involved in a game to form groups of groups. Consequently, we can expect self-interested individuals (and groups) to accept losses in one level of the multi-level structure for gains in another level. This type of tradeoff plays a central role in demonstrating the complicated nature of stability. In particular,

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when different levels of group formation are considered simultaneously, we get the following two phenomena: (1) overall stability of the structure can arise from unstable levels, and (2) stable groups can be destabilized when players make tradeoffs across levels.

While an illustration of point (1) must wait until the next section, the Iraq Coalition Government, provides a convenient illustration of point (2). There are serious game theoretic considerations that complicate the creation of a coalition government from the Sunnis, Shias, and Kurds. Military, religious and political scholars all note that cooperation among the ethnic groups is not guaranteed, yet it seems to have been missed that the imposition of a coalition government among these groups might cause the individual groups to dissolve. That is, now that there exists a power sharing game, there might be an alliance among subsets of the Sunnis, Shias and Kurds in which members benefit from joining while excluding other members of their ethnic group. The tradeoff for political power may destabilize what appears to be a stable ethnic group structure.

1.1 Coral Games and Coalitional Games

Neither a single characteristic function form game [see von Neumann (1928); Shapley (1967); Scarf (1967); Aumann and Dreze (1974)] nor a single partition function form game [see Thrall and Lucas (1963); Bloch (1996); Yi (1997); Ray and Vohra (1999)] can capture the tradeoffs and complexities that arise in multi-level structures. Each approach describes group formation given an exogenous set of players. However, in a multi-level structure, the set of players in a game among groups is not exogenously given but rather depends on payoff maximizing tradeoffs across levels. The coral games framework uses characteristic and/or partition functions in a more general model that allows for co-dependence among levels.

This paper develops an equilibrium concept for coral games called the core of cores[1]. Results describe the way in which the stability of each level depends on the other levels. These include necessary and sufficient conditions for the existence of the core of cores, several corollaries on the properties of the core of cores, and a theorem that establishes a relationship between the cores of each component game and the core of cores of the multi-level structure.

Other authors have looked at similar ways of enriching coalitional structures. Derks and Gilles (1995) and Demange (2004) examine the effect of restrictions that arise from a hierarchical structure on coalition formation. Not to be confused with the multi-level structures discussed in the current paper, these papers refer to a hierarchical permission structure to capture the relationship between superiors and subordinates. Demange argues that permission structures can allow groups to organize efficiently when such organization is otherwise unstable.

In a theme that is very closely related to the model presented below, Winter (1989) develops the idea of a "levels structure" wherein a coalition can be thought to have a type of substructure, each level of which consists of a cooperative agreement. Winter gives a value for such structures and shows that this value is an extension of the Shapley value and others. However, he stops short of allowing for the endogeneity of these substructures, and hence does not develop an equilibrium concept.

Perhaps the most closely related work is done in a series of papers by Herings, Van Der Laan,

[1]The term, core of cores, was suggested by Donald G. Saari

and Talman (2003, 2007a,b) regarding "socially structured games." Like other coalitional models, their model involves a set of players partitioning into coalitions. However, the innovation comes in that each coalition has an internal (social) structure. This social structure is driven by the exogenously specified power of the players. The power of a player determines the degree to which that player may take payoffs from less powerful players subject to the restriction that the less powerful players cannot do better by forming another internal organization in which they can secure higher payoffs. Hence the socially structured game needs a new equilibrium concept, the *socially structured core*, which the authors show is a subset of the core of the game without social structure. Although Herings et al. allow the internal structure to be endogenous, they stop short of allowing for the internal structure to involve separate (yet dependent) payoffs as in the coral games model.

The results also suggest a new way of thinking about dynamic group formation that is complementary to the extensive work on the dynamics of coalition formation [see Hart and Kurz (1983); Ray and Vohra (1999); Konishi and Ray (2003); Hyndman and Ray (2006); Manea (2007)]. In addition to looking at the dynamics of bargaining and multi-lateral deviations as is well-covered by the literature, one can also consider the dynamics of sequential formation of multi-level structures. If the multi-level structure forms sequentially, we have that individuals form groups, then groups form groups, and so on. In the core of cores, a multi-level structure can be stable even though some (or all) of its levels are independently unstable. Hence, a stable structure may be unlikely to form if the first levels in the sequence are unstable.

1.2 Coral Games and Political Economy

The field of political economy often involves situations where individuals and groups are forced to balance tradeoffs between multiple activities. Researchers in this area have long known that such environments can create equilibria that are not the sum of their parts. In other words, optimal behavior for the overall situation is not necessarily comprised of equilibrium behavior in each of the individual components that constitute the overall environment. The work of Tsebelis (1991) on nested games captures well the idea of co-dependence in noncooperative voting games. He documents the way in which voting coalitions form to optimize with respect to a voting process that involves multiple stages. The behavior in each stage is not an equilibrium of that stage but is a component of optimal behavior for the overall voting process.

Many scholars have applied this idea of co-dependent games to find a stabilizing effect in governance and constitutional design. Hammond and Miller (1987) discuss the stabilizing effects of bicameralism and the executive veto on the legislative process and the ability of the legislative override to counter that effect. Krehbiel (1988), Tsebelis and Money (1997) and Tsebelis (2000) find similar stabilizing effects of bicameralism. The key in many of these settings is the way co-dependent games limit profitable deviations by imposing unacceptable tradeoffs.

Noh (2002) and Garfinkel (2004) provide two more political economy applications by analyzing resource distribution within a coalition in the presence of conflict with other coalitions. Because they consider only the payoffs from the conflict with other coalitions, their analysis adheres closely to the literature on endogenous coalition formation. However, it suggests strongly the importance of a multi-level analysis. One of the advantages of the current model is to capture the conflict among coalitions while simultaneously allowing for the "within-group" interaction to generate payoffs. Hence, the coral games framework deals with both the multi-level nature of organizational structures and multi-dimensional payoffs in a straightforward manner.

The outline of the paper is as follows. First, the coral games framework is detailed for the case of transferable payoffs and two levels. It is then extended to include non-transferable payoffs as well as an arbitrary number of levels. Finally, a simple example involving political party formation and interaction (inspired by Winter's example) provides a concrete illustration of several theoretical results, including the existence theorem and its corollaries.

2 Definitions and Results

2.1 Transferable Utility on Two Levels

In this section, the coral games framework is formally defined for games that award transferable payoffs on two levels. The payoffs on each level can come either from a partition function or from a characteristic function; the form need not be the same on both levels. In the formal model below, it is assumed that level one uses a characteristic function and level two uses a partition function. This is done in order to demonstrate the versatility of the coral games approach and to make clear the opening remark. It should be noted, however, that the definitions are easily adapted to handle all combinations of characteristic and partition function payoffs with only slight changes in notation.

Assume that there are N players and that their interaction takes place on two levels. On level one they are organized into a partition $\mathcal{P} = \{P_1, P_2, ...\}$; i.e. $N = \cup_i P_i$ and $P_i \cap P_j = \emptyset$ for $i \neq j$. A partition $\mathcal{P}$ is an element of $\Omega(N)$, the set of all partitions of N. The elements of $\mathcal{P}$ are coalitions of players. The payoff available to a generic subset of the players, $S \subseteq N$, is given by a characteristic function $v : \mathcal{C} \to \mathbb{R}$, where $\mathcal{C}$ is the set of all nonempty subsets of N. Thus the payoff to the ith element of $\mathcal{P}$ is $v(P_i)$.

On level two, the elements of $\mathcal{P}$ form a partition $\mathcal{B} = \{B_1, B_2, ...\}$, where $\mathcal{B}$ is in $\Omega(\mathcal{P})$, the set of partitions of $\mathcal{P}$. To be clear, $\mathcal{B}$ is a partition of $\mathcal{P}$; it is a partition of coalitions. The elements of $\mathcal{B}$ are coalitions of coalitions. $B_k = \{P_i, P_j, ...\}$ has a single payoff that is to be divided among its member coalitions, P_i, and again within each coalition. This payoff is given by the function, $w : \mathcal{B} \times \Omega(\mathcal{P}) \longrightarrow \mathbb{R}$. Note that $w(\cdot, \cdot)$ is defined for all $\mathcal{P} \in \Omega(N)$.

The N players are assumed to have preferences over the payoffs they receive from levels one and two. Let $U_j(x(j), y(j))$ represent player j's utility, where $x(j)$ is the payoff that player j receives from level one and $y(j)$ is the payoff that player j receives from level two. These payoffs must be feasible, i.e. $\sum_{j \in P_i} x(j) \leq v(P_i)$ and $\sum_{j \in B_k} y(j) \leq w(B_k, \mathcal{B})$. To simplify the mathematics, assume preferences are convex and the level sets are globally invertible so we can write $x_j(y)$ and $y_j(x)$. In other words, given a level of utility, $\bar{U}$, and the payoffs from level one, x, we can find the payoffs from level two, y, that solve $U_j(x, y) = \bar{U}$ (likewise in the other direction). This brings us to the definition of a coral game.

Definition 2.1. A **co-dependent organizational levels** (or **coral**) **game in transferable payoffs** is given by $\langle N, v, w, \succsim \rangle$, where N is the number of players, $v(\cdot)$ gives the payoffs on level one, $w(\cdot, \cdot)$ gives the payoffs on level two, and $\succsim$ represents preferences over payoffs (x, y).

As stated in the introduction and enforced through the development of the formal model, characteristic and partition function form games are special cases of coral games.

Remark. Characteristic and partition function games are special cases of coral games.

1. A characteristic function game on level one is a coral game in which the players are N, $v(P_j)$ is a characteristic function and $w(B_k, \mathcal{B}) = 0$ for all $B_k \in \mathcal{B}$ and all $\mathcal{B} \in \Omega(\mathcal{P})$.
2. A partition function game on level two is a coral game in which the players are elements of a fixed $\mathcal{P}$, $w(B_k, \mathcal{B})$ is a partition function and $v(P_j) = 0$ for all $P_j \in \mathcal{P}$. This game is written as $\langle \mathcal{P}, w \rangle$, the partition function game in which the players are fixed to be the elements of $\mathcal{P}$.

The idea is that a coral game is a generalization of existing coalitional games. In coalitional games, one of two strong assumptions is usually made. Namely, either the only interaction is between individuals (characteristic function games), or the interaction is between coalitions. The following results show how insight is gained by fitting applications of characteristic and partition function form games into the coral games framework. In a basic way, then, these results explain the existing literature as cross-sections of larger phenomena.

As mentioned in the introduction, this paper introduces a solution concept, the core of cores, to accompany the coral games model. The core of cores is similar to the traditional concept of the core. It is an allocation, and an associated group structure, such that no multilateral deviations are beneficial. However, since each coalition $B_k \in \mathcal{B}$ is comprised of one or more subcoalitions $P_j \in \mathcal{P}$, deviating coalitions must have the same structure. In particular, a deviating coalition is a subset of players $S \subset N$ organized as a structure $\mathcal{S}$ that is comprised of one or more subcoalitions $S_j \in \mathcal{S}$. Hence, the core of cores requires that no group S can organize themselves in a way that allows every member to improve his or her overall welfare, where overall welfare is a result of organization and interaction on both levels.

Definition 2.2. $(x, y, \mathcal{P}, \mathcal{B})$ is in the **core of cores** of $\langle N, v, w, \succsim \rangle$ if $\nexists S \subset N$, with partition $\mathcal{S} = \{S_1, S_2, ...\}$, and $(x', y', \mathcal{P}', \mathcal{B}')$ such that $(x', y') \succ_S (x, y)$, $\sum_{j \in S_i} x'(j) = v(S_i)$ for all $S_i \in \mathcal{S}$, $\sum_{j \in S} y'(j) = w(S, \mathcal{B}')$, $\mathcal{P}' = \{\mathcal{P} \setminus S, \mathcal{S}\}$, and $\mathcal{B}' = \{\mathcal{B} \setminus S, S\}$.

In order to state the next result, an *excess* function for coral games must first be defined.

Definition 2.3. For a coral game $\langle N, v, w, \succsim \rangle$ and allocations $(x, y, \mathcal{P}, \mathcal{B})$ let the **excess**, $e^*(\mathcal{S}, x, y, \mathcal{P}, \mathcal{B})$, for a partition $\mathcal{S}$ of a subset of players, $S \subset N$, be the constrained maximized value of $e(\mathcal{S}, x, y, \mathcal{P}, \mathcal{B})$ with respect to x', where

$$e(\mathcal{S}, x, y, \mathcal{P}, \mathcal{B}) = [w(S, \mathcal{B}') - y(S)] - \left[\sum_{j \in S} \int_{x(j)}^{x'(j)} \frac{dy_j(x)}{dx} dx \right] \tag{2.1}$$

and the constraints are

$$\sum_{j \in S_i} x'(j) \leq v(S_i) \quad \forall S_i \in \mathcal{S} \tag{2.2}$$

$$x'(j) \geq 0, \tag{2.3}$$

$$y_j(x'(j)) \geq 0, \quad \forall j \in S \tag{2.4}$$

and where $\mathcal{B}' = \{\mathcal{B} \setminus S, S\}$ and $y_j : \mathbb{R} \longrightarrow \mathbb{R}$ defines the level set $\bar{U} = U_j(x, y)$. Thus $\frac{dy_j(x)}{dx}$ is player j's marginal rate of substitution between level two and level one payoffs when $\bar{U} = U_j(x, y)$.

The excess function is the sum of the level one and level two gains and losses accrued to each S_i by deviation with the coalition of coalitions $\mathcal{S}$. We use the marginal rate of substitution between level one and level two payoffs in order to measure level one gains (or losses) in terms of level two payoffs. When the extra arguments are obvious, notation is simplified from $e(\mathcal{S}, x, y, \mathcal{P}, \mathcal{B})$ to $e(\mathcal{S})$.

The three constraints are straightforward. The first is the budget constraint for level one payoffs. The second simply says that level one payoffs cannot be negative. The third states that player j cannot offer more level two payoffs to group $\mathcal{S}$ than she currently has. Note that in an environment where players have sufficient reserves, constraints 2.3 and 2.4 are not necessary. However, the current analysis does not allow for borrowing.

With more inequality constraints $(2|S| + |\mathcal{S}|)$ than free variables $(|S|)$, we need to know which are binding before we can find $e^*(\mathcal{S})$. Therefore, the following algorithm is useful for solving the above maximization problem. In finding the maximum, the algorithm identifies which of the $2|S| + |\mathcal{S}|$ constraints are binding. But first, some additional notation must be introduced. N^t is the set of players remaining after round $t-1$, $N^{t,\prime}$ is the set of players eliminated in round $t-1$, and $v^t(S_i) = v^{t-1} - \sum_{j \in N^{t,\prime}} x'(j)$.

For each $S_i \in \mathcal{S}$,

1. Maximize $e(\mathcal{S}, x, y, \mathcal{P}, \mathcal{B})$ subject to $\sum_j x(j) \leq v(S_i)$ to get x^1.
2. For $j \in S_i$ check constraints 2.3 and 2.4.
   (a) if $y_j(x^t) < 0$ set $x'(j)$ to solve $y_j(x) = 0$.
   (b) if $x^t(j) < 0$ set $x'(j) = 0$.
   (c) otherwise $x^t(j) = x'(j)$.
3. To get x^{t+1}
   (a) If $\sum_j [x^t(j) - x'(j)] < 0$,
      i. $N^{t+1,\prime} = \{j : x^t(j) = 0\}$
      ii. For $j \in N^{t+1,\prime}$, fix $x^*(j) = 0$.
      iii. Maximize $e(\mathcal{S}, x, y, \mathcal{P}, \mathcal{B})$ over $x^{t+1}(j)$ for $j \in N^{t+1}$ subject to
      $$\sum_{j \in N^{t+1}} x^{t+1}(j) \leq v^{t+1}(S_i).$$
      iv. Return to step 2.
   (b) If $\sum_j [x^t(j) - x'(j)] > 0$,
      i. $N^{t+1,\prime} = \{j : y_j(x^t) = 0\}$
      ii. For $j \in N^{t+1,\prime}$, fix $x^*(j) = x'(j)$.

iii. Maximize $e(\mathcal{S}, x, y, \mathcal{P}, \mathcal{B})$ over $x^{t+1}(j)$ for $j \in N^{t+1}$ subject to

$$\sum_{j \in N^{t+1}} x^{t+1}(j) \leq v^{t+1}(S_i).$$

iv. Return to step 2.

(c) If $\sum_j [x^t(j) - x'(j)] = 0$ fix $x^*(j) = x'(j)$ and terminate.

Proposition 1. This algorithm converges to the maximum in at most $max\{|S_i| : S_i \in \mathcal{S}\}$ iterations.

Proof. We write the Lagrange multiplier from the constrained maximization to attain x^t for S_i, as λ_i^t. Suppose the algorithm ends in time τ, then the conditions for x^* to maximize the excess equation are:

- if $\frac{dy_j(x^*)}{dx} > \lambda_i^\tau$ then $y_j(x^*) = 0$
- if $\frac{dy_j(x^*)}{dx} < \lambda_i^\tau$ then $x^* = 0$
- otherwise $\frac{dy_j(x^*)}{dx} = \lambda_i^\tau$
- $\sum_{j \in N} x^*(j) = v(S_i)$ or $y_j(x^*) = 0$ for all $j \in N$.

Now we make two claims that will be necessary and sufficient to show that the algorithm converges to the maximum:

1. If $\lambda_i^{t+1} < \lambda_i^t$, then $\lambda_i^{t+k} < \lambda_i^t$ for all positive integers $k \geq 2$.
2. If $\lambda_i^{t+1} > \lambda_i^t$, then $\lambda_i^{t+k} > \lambda_i^t$ for all positive integers $k \geq 2$.

Starting with the first claim. Without loss of generality, assume $t = 0$ to simplify notation. Now suppose $\lambda_i^1 < \lambda_i^0$. Then

$$\sum_{j \in N^1} x^1(j) < \sum_{j \in N^1} x^0(j)$$

because

$$\begin{aligned} \sum_{j \in N^0} x^0(j) &= v^0(S_i), \text{ and} \\ \sum_{j \in N^1} x^1(j) &= v^1(S_i) \end{aligned}$$

where $N^0 \supseteq N^1$ and $v^0(S_i) > v^1(S_i)$. In other words, to get N^1 we removed players from N^0 with $x^0(j) < 0$.

In order to show that $\lambda_i^k < \lambda_i^0$ we will show that

$$\sum_{j \in N^k} x^k(j) \leq \sum_{j \in N^k} x^0(j). \tag{2.5}$$

Rewriting each side of the the equation we get

$$\begin{aligned}\sum_{j\in N^k} x^k(j) &= v^0(S_i) - \sum_{m=1}^{k}\sum_{j\in N^{m,\prime}} x'(j), \text{ and}\\ \sum_{j\in N^2} x^0(j) &= v^0(S_i) - \sum_{m=1}^{k}\sum_{j\in N^{m,\prime}} x^0(j).\end{aligned}$$

We can rewrite 2.5 as:

$$\sum_{m=1}^{k}\sum_{j\in N^{m,\prime}} \left[x^0(j) - x'(j)\right] \le 0$$

or alternatively as

$$\sum_{m=2}^{k}\sum_{j\in N^{m,\prime}} \left[x^0(j) - x'(j)\right] \le \sum_{j\in N^{1,\prime}} \left[x'(j) - x^0(j)\right]. \tag{2.6}$$

However, since $\lambda_i^1 < \lambda_i^0$, we know that $x'(j) = 0$ for all $j \in N^{1,\prime}$. Together with the facts:

$$\begin{aligned}\sum_{m=2}^{k}\sum_{j\in N^{m,\prime}} x^0(j) &= v^0(S_i) - \sum_{j\in N^1} x^0(j) - \sum_{j\in N^k} x^0(j) \text{ and}\\ -\sum_{m=2}^{k}\sum_{j\in N^{m,\prime}} x'(j) &= v^k(S_i) - v^1(S_i),\end{aligned}$$

we can rewrite inequality 2.6 as

$$\begin{aligned}v^0 - \sum_{j\in N^1} x^0(j) - \sum_{j\in N^k} x^0(j) + v^k - v^1 &\le -\sum_{j\in N^{1,\prime}} x^0(j)\\ v^0 - \sum_{j\in N^1} x^0(j) - \sum_{j\in N^k} x^0(j) + \sum_{j\in N^{1,\prime}} x^0(j) &\le v^1 - v^k.\end{aligned}$$

Since $v^0(S_i) = v^1(S_i)$, we have

$$v^k \le \sum_{j\in N^1} x^0(j) + \sum_{j\in N^k} x^0(j) - \sum_{j\in N^{1,\prime}} x^0(j).$$

This always holds because $x^0(j) < 0$ for all $j \in N^{1,\prime}$ and $x^0(j) > 0$ for all $j \in N^1$ and N^k.

The proof of the second claim is exactly the inverse argument of the first, omitted here to conserve space. Together, they guarantee that the conditions for a maximum are met, because λ_i^k is either $\max\{\lambda_i^t\}$, $\min\{\lambda_i^t\}$ for $t < k$, or in $(\lambda_i^{k-1}, \lambda_i^{k+1})$.

To see that the algorithm converges to the maximum in $\tau \le \max\{|S_i|\}$ rounds, consider the following argument. For every round, either $\sum_{j\in N^t} x^t(j) = 0$ or the algorithm continues to the next step. When the algorithm continues, some subset of the players $N^{t+1,\prime}$ have $x^*(j)$ fixed at $x'(j)$. Hence, in every round that the algorithm continues, there are strictly fewer free variables $x(j)$ than in the previous round. In the extreme, if $|N^{t,\prime}| = 1$ for every round $t < \tau$, then the routine only lasts as long as there are members of the largest group S_i. $\square$

Having established a method for computing the excess, the next result uses $\mathcal{S}$'s excess, $e^*(\mathcal{S})$, to give a necessary and sufficient condition for payoffs (x, y) and structure $(\mathcal{P}, \mathcal{B})$ to be in the core of cores.

Theorem 1. *$(x, y, \mathcal{P}, \mathcal{B})$ is in the core of cores of $\langle N, v, w, \succsim\rangle$ if and only if $e^*(\mathcal{S}) \leq 0$ for every partition, $\mathcal{S}$, of every subset $S \subset N$.*

The proof is provided for a more general statement of this theorem in section 3.2.

Theorem 1 is best understood by thinking about the tradeoff between the two levels. There can be no tradeoff available to any subset of the players that results in a pareto improvement for the members of that subset. By converting gains on level two into level one payoffs, we know how much level two must compensate for losses on level one or vice versa. As the excess function for standard coalitional games specifies how much coalition S forgoes to be in the grand coalition (or partition structure), so does the excess in coral games. However, the difference is that this version makes corrections for the more complicated structure of coral games, including the tradeoffs between different types of payoffs. In addition, the excess in coral games easily incorporates nontransferable payoffs on level one. Further interpretation of theorem 1 comes in the form of three corollaries.

Corollary 1. *Suppose $(x, \mathcal{P}) \in Co(N, v)$ and $(y, \mathcal{B}) \in Co(\mathcal{P}, w)$ where $\mathcal{B}$ is a partition of $\mathcal{P}$. Then $(x, y, \mathcal{P}, \mathcal{B})$ is not necessarily in $Co(N, v, w, \succsim)$, the core of cores.*

Proof. This follows from the endogeneity of $\mathcal{P}$ and the fact that payoffs on level two, $w(B_k, \mathcal{B})$, are defined for all $\mathcal{P} \in \Omega(N)$. The game $\langle \mathcal{P}, w\rangle$ takes the players, elements of $\mathcal{P}$, as given whereas in a coral game that set is endogenous. Therefore, a coalition S might be able to reorganize as $\mathcal{S}$ so that

$$e^*(S) = [w(S, \mathcal{B}') - y(S)] - \left[\sum_{j \in S} \int_{x(j)}^{x'(j)} \frac{dy_j(x)}{dx} dx\right] > 0.$$

$\mathcal{B}'$ is not a partition of $\mathcal{P}$ but rather is a partition of $\mathcal{P}' = \{\mathcal{P} \setminus S, \mathcal{S}\}$. Hence we can have that $w(S, \mathcal{B}')$ is greater than $y(S)$ even though $(y, \mathcal{B}) \in Co(\mathcal{P}, w)$. $\square$

Corollary 1 shows that the sum of stable levels is not a stable whole. Even when level one is stable and level two is stable given the partition that results from level one, the overall structure can be unstable as a result of the endogeneity of each partition. The next result shows that a stable whole can be the sum of unstable parts.

Corollary 2. *$(x, y, \mathcal{P}, \mathcal{B}) \in Co(N, v, w, \succsim)$ does not imply*

1. *$(x, \mathcal{P}) \in Co(N, v)$, or*

2. $(y, \mathcal{B}) \in Co(\mathcal{P}, w)$.

Proof. Suppose there is some structure $\mathcal{P}' = \{\mathcal{P} \setminus S, \mathcal{S}\}$ on level one such that $v(S_i) - x(S_i) > 0$ for all $S_i \in \mathcal{S}$. Then $(x, \mathcal{P})$ is not in $Co(N, v)$. However, depending on the game among coalitions, S can suffer losses on level two by rearranging into $\mathcal{P}'$. Equation 2.1 says that S will not deviate from $\mathcal{P}$ if the losses in level two payoffs outweigh the gains in level one payoffs.

A similar argument can be made in the other direction. Suppose that by breaking from the structure $\mathcal{B}$ on level two, a subset S, arranged as $\mathcal{S}$, has that $w(S, \mathcal{B}') - y(S) > 0$. Then, $(y, \mathcal{B})$ is not in $Co(\mathcal{P}, w)$. However, depending on the game within coalitions, S might suffer losses on level one by rearranging into $\mathcal{P}'$. Equation 2.1 says that S will not deviate from $\mathcal{B}$ if the losses in level one payoffs outweigh the gains in level two payoffs.

Finally, these scenarios can exist simultaneously. They can even exist for the same $S \subset N$ as long as they are for different partitions $\mathcal{S}$. To be clear, it is possible that a partition of a subset, $\mathcal{S}$, can do strictly better on level one by deviating from $\mathcal{P}$, yet suffers from this deviation with even greater losses on level two. At the same time, we can have a different partition of S, $\mathcal{S}'$, that does strictly better on level two by deviating from $\mathcal{B}$, yet suffers from this deviation with even greater losses on level one. Hence, we would have a situation in which $(x, \mathcal{P})$ is not in $Co(N, v)$ and $(y, \mathcal{B})$ is not in $Co(N, w)$, yet $(x, y, \mathcal{P}, \mathcal{B})$ is still in the core of cores of $\langle N, v, w, \succsim \rangle$. $\square$

Corollary 2 says that we do not need the levels to be individually stable in order to create a stable multi-level structure. This is a powerful insight. Many times our analysis reveals an empty core, yet the real world does not always bear out such instability. Corollary 2 suggests that this can be due to another level's influence on the unstable game, creating stability overall.

Often we have a situation, such as one might speculate is happening in the Iraq coalition government, where a coalitional game is imposed among the coalitions of an existing structure. In this case, we want to know what effect the game among coalitions will have on the stability of the coalitions themselves. The following result provides a condition under which the pre-existing coalition structure will be unstable.

Corollary 3. *If $(x, \mathcal{P}) \in Co(N, v)$ then $(x, y, \mathcal{P}, \mathcal{B})$ is not in $Co(N, v, w, \succsim)$ if there exists $\mathcal{S}$ and x^* for every $(y, \mathcal{B})$ such that*

$$w(S, \mathcal{B}') > \sum_{j \in S} y_j(x^*).$$

Proof. For the purposes of theorem 1 and its proof in section 3, it is useful to see the entire excess function and the way that it balanced net gains on both levels against each other. However, we can simplify equation 2.1. First break the term

$$\sum_{j \in S} - \int_x^{x^*} \frac{dy_j(x)}{dx} dx$$

into

$$\sum_{j \in S} -y_j(x^*) + y_j(x)$$

where $y_j(x)$ is simply the sum of the anti-derivatives of $\frac{dy_j(x)}{dx}$ evaluated at x. Then, because $\frac{dy_j(x)}{dx}$ is defined implicitly for $\bar{U}_j = U_j(x, y)$, this means that

$$\sum_{j \in S} y_j(x) = y(S).$$

The excess for coalition S is then

$$e^*(\mathcal{S}) = w(S, \mathcal{B}') - \sum_{j \in S} y_j(x^*).$$

Written this way, S's excess is the difference between its payoffs on level two, controlling for the effect of the change in level one payoffs.

From theorem 1 we know that the profile of payoffs and structures $(x, y, \mathcal{P}, \mathcal{B})$ is not in the core of cores if $e^*(\mathcal{S}) > 0$ for some $\mathcal{S}$. Therefore, if there is no $(y, \mathcal{B})$, where $\mathcal{B}$ is a partition of $\mathcal{P}$, such that theorem 1 is satisfied, then $(x, \mathcal{P})$ is not a part of the core of cores. □

Corollary 3 introduces a straightforward way of looking at the dynamic formation of organizational structures. It determines when level two destabilizes level one. Also, to put this in the context of the literature, consider the criticism of characteristic function form games that they ignore the interaction between coalitions. By using the coral games approach it is now clear when this is a safe assumption and when it is not. According to corollary 3, when the payoffs generated by the interaction between coalitions present strong enough tradeoffs, it is not a safe assumption.

But there is a corresponding criticism of partition function form games regarding the payoffs from interaction within each coalition. Sometimes the payoffs from interaction within each coalition are included in the partition function itself. That approach is unsatisfactory because it does not allow us to see the way the levels are connected. If we do not see the way they are connected, then we do not know whether an organizational structure can be built sequentially; first by the formation of coalitions, then by the formation of coalitions of coalitions as in corollary 3.

The other option when dealing with partition function games is to assume that the interaction among coalitions is the only relevant aspect of the game. Given our understanding of coalitions and corollary 3, we should not expect that this assumption will always hold. So it is important to reexamine our approach to partition function games, taking care not to overlook the effect that other levels in the structure have on the stability of the level in question.

The above results, particularly corollary 2, established that the core of cores is not limited in its construction to stable components, and that it can arise from any combination of stable and unstable levels. However, it is often quite easy to construct mathematical coincidences, such as three lines intersecting in two dimensions. So we need to know if a core of cores consisting of unstable components is a rarity or if it generically exists. One way of doing so is by looking at a core of cores that consists of stable components $(x, \mathcal{P})$ and $(y, \mathcal{B})$, then proceeding to look at perturbations of the functions v and w such that neither the first nor the second level is stable, yet they are stable together. Here there is an open set of perturbed functions, (v', w'), to satisfy that condition.

Theorem 2. *For a coral game in transferable payoffs $G = \langle N, v, w, \succsim \rangle$ where $N \geq 3$, $(x, \mathcal{P}) \in Co(N, v)$, $(y, \mathcal{B}) \in Co(\mathcal{P}, w)$, $|\mathcal{P}| \geq 2$ and $\mathcal{B}$ is a partition of $\mathcal{P}$, so that $(x, y, \mathcal{P}, \mathcal{B})$ is in the core of cores of G, there exists an open set of functions*

$$\{(v', w') : \exists\ (x', \mathcal{P}) \notin Co(N, v'),\ (y', \mathcal{B}) \notin Co(\mathcal{P}, w'),\ \textit{and } (x', y', \mathcal{P}, \mathcal{B}) \in Co(N, v', w', \succsim)\}$$

Proof. This result is proven by construction. First select vectors α and β that satisfy the following conditions:

$$0 \leq \alpha(j) \leq x(i)$$
$$0 \leq \beta(j) \leq y(j)$$

where $\alpha(j)$ and $\beta(j)$ are the jth elements of the respective vectors. With α and β we then perturb $v(P_i)$ and $w(B_k, \mathcal{B})$ in the following manner:

$$v'(P_i) = v(P_i) - \alpha(P_i) - \sum_{j \in S_w \cap P_i} \int_{\underline{y}}^{\bar{y}} dx_j(y), \quad \forall P_i \in \mathcal{P} \tag{2.7}$$

$$w'(B_k, \mathcal{B}) = w(B_k, \mathcal{B}) - \beta(B_k) - \sum_{j \in S_v \cap B_k} \int_{\underline{x}}^{\bar{x}} dy_j(x), \quad \forall B_k \in \mathcal{B}. \tag{2.8}$$

$w(S, \mathcal{B})$ becomes:

$$w'(S, \mathcal{B}) = \begin{cases} w(S, \mathcal{B}) - \Delta & \text{if } \mathcal{S} \cap \mathcal{P} \neq \varnothing \\ w(S, \mathcal{B}), & \text{otherwise} \end{cases}$$

where $\alpha(S) = \sum_{j \in S} \alpha(j)$, $\beta(S)$ is likewise defined, and

- $\bar{x} = x_j\,(y(j) - \beta(j))$,
- $\underline{x} = x_j\,(y(j) - \beta(j)) - \alpha(j)$,
- $\bar{y} = y(j)$, and
- $\underline{y} = y(j) - \beta(j)$.

We choose α and β so that:

$$e(S_v, x) = v'(S_v) - x(S_v) + \alpha(S_v) + \sum_{j \in S_v \cap S_w} \int_{\underline{y}}^{\bar{y}} dx_j(y) > 0$$

$$e(S_w, y, \mathcal{B}) = w'(S_w, \mathcal{B}') - y(S_w) + \beta(S_w) + \sum_{j \in S_w \cap S_v} \int_{\underline{x}}^{\bar{x}} dy_j(x) > 0$$

for some sets S_v and S_w. Since $e(S_v, x) = 0$ when $S_v = P_i$ for some $P_i \in \mathcal{P}$ and $e(S_w, y) = 0$ when $S_w = B_k$ for some $B_k \in \mathcal{B}$, these will be restrictions on S_v and S_w. Now the following allocations are feasible under v' and w':

$$x'(j) = x(j) - \alpha(j) - \int_{\underline{y}}^{\bar{y}} dx_j(y)$$

$$y'(j) = y(j) - \beta(j) - \int_{\underline{x}}^{\bar{x}} dy_j(x).$$

First we show that $(x', y', \mathcal{P}, \mathcal{B})$ is in the core of cores of the perturbed game $G' = \langle N, v', w', \succsim \rangle$. Consider the excess for a coalition S partitioned as $\mathcal{S}$:

$$e^*(\mathcal{S}, G', x', y') = w'(S, \mathcal{B}) - y'(S) - \sum_{j \in S} \int_{x'(j)}^{x'^*} dy_j(x).$$

Here we can break $y'(S)$ down as follows

$$y'(S) = y(S) - \beta(S) - \sum_{j \in S} \int_{\underline{x}}^{\bar{x}} dy_j(x).$$

Note also that

$$\int_{x'(j)}^{x'^*} dy_j(x) = \int_{x(j)}^{x'^*} dy_j(x) + \int_{x'(j)}^{x(j)} dy_j(x).$$

Then $e^*(\mathcal{S}, G', x', y')$ becomes

$$\begin{aligned} e^*(\mathcal{S}, G', x', y') &= w'(S, \mathcal{B}) - y(S) + \beta(S) \\ &\quad + \sum_{j \in S} \left[\int_{\underline{x}}^{\bar{x}} dy_j(x) - \int_{x'(j)}^{x(j)} dy_j(x) \right] \\ &\quad - \sum_{j \in S} \int_{x(j)}^{x'^*} dy_j(x). \end{aligned}$$

Next we show that

$$\beta(S) = \sum_{j \in S} \left[\int_{x'(j)}^{x(j)} dy_j(x) - \int_{\underline{x}}^{\bar{x}} dy_j(x) \right]. \tag{2.9}$$

To do so, we first demonstrate that $\underline{x} = x'(j)$:

$$\begin{aligned} x_j\left(y(j) - \beta(j)\right) - \alpha(j) &= x(j) - \alpha(j) - \int_{\underline{y}}^{\bar{y}} dx_j(y) \\ x_j\left(y(j) - \beta(j)\right) - x(j) &= -\int_{\underline{y}}^{\bar{y}} dx_j(y). \end{aligned}$$

where $x_j(\underline{y}) = x_j(y(j) - \beta(j))$ and $x(j) = x(\bar{y})$. This means that equation 2.9 becomes

$$\beta(S) = \sum_{j \in S} \left[\int_{\bar{x}}^{x(j)} dy_j(x) \right].$$

Recall that $\bar{x} = x_j(y(j) - \beta(j))$. Therefore we have

$$\begin{aligned} \beta(S) &= \sum_{j \in S} [y(j) - y(j) + \beta(j)] \\ &= \sum_{j \in S} \beta(j). \end{aligned}$$

Finally, we are left with

$$e^*(\mathcal{S}, G', x', y') = w'(S, \mathcal{B}) - y(S) - \sum_{j \in S} \int_{x}^{x'^*} dy_j(x).$$

If $\mathcal{S} \cap \mathcal{P} = \emptyset$ then this is identical to $e^*(\mathcal{S}, G, x, y)$. Thus we have that $x'^* = x^*$ for every $\mathcal{S}$. And because $(x, y, \mathcal{P}, \mathcal{B})$ is in the core of cores of G, then $(x', y', \mathcal{P}, \mathcal{B})$ is in the core of cores of the perturbed game G'.

If $\mathcal{S} \cap \mathcal{P} \neq \emptyset$ then we have

$$e^*(\mathcal{S}, G', x', y') = w(\mathcal{S}, \mathcal{B}) - \Delta - y(S) - \sum_{j \in S} \int_{x}^{x'^*} dy_j(x).$$

The fact there are P_i in $\mathcal{S}$ changes the constraints. Instead of $\sum_{j \in S_i} x'^* = v(S_i)$ for all $S_i \in \mathcal{S}$, we have $\sum_{j \in S_i} x'^* = v'(P_i)$ for some $S_i \in \mathcal{S}$. And since we allow $v'(P_i)$ be strictly greater than $v(P_i)$, our perturbation must account for this. So we look for an upper bound on the difference:

$$D(x'^*, x^*) = -\sum_{j \in S} \int_{x}^{x'^*} dy_j(x) + \sum_{j \in S} \int_{x}^{x^*} dy_j(x) = -\sum_{j \in S} \int_{x^*}^{x'^*} dy_j(x).$$

To attain this upper bound, we assume neither $x_j'^* \geq 0$ nor $y_j(x'^*) \geq 0$ are binding for any $j \in P_i \in \mathcal{S}$. Hence $D(x'^*, x^*)$ is maximized. Then it follows that

$$v'(P_i) - v(P_i) = -\alpha(P_i) - \sum_{j \in P_i} \int_{\underline{y}}^{\bar{y}} dx(y).$$

Using the convexity of preferences, we get an upper bound:

$$\Delta = \sum_{P_i \in \mathcal{S}} \left[\frac{v'(P_i) - v(P_i)}{|P_i|} \right] \sum_{j \in P_i} \left(\frac{-dy_j(x^*)}{dx} \right).$$

We must show that this implies $w'(S, \mathcal{B}') \geq 0$, and when it is written out, it becomes

$$w(\mathcal{S}, \mathcal{B}') - \sum_{P_i \in \mathcal{S}} \left[\sum_{j \in P_i} \left(\frac{-dy_j(x^*)}{dx} \right) \left[-\alpha(j) - \int_{\underline{y}}^{\bar{y}} dx_j(y) \right] \right] \geq 0.$$

It can be rewritten as:

$$(-\tilde{\alpha} - \tilde{x}(y) + \tilde{x}(y - \beta)) \cdot -\nabla \tilde{y}(x^*) \leq w(\mathcal{S}, \mathcal{B}').$$

where $\tilde{x}$ is the vector with elements $x(j)$ indexed by $j \in P_i$ for each $P_i \in \mathcal{S}$, $\tilde{x}(y - \beta)$ and $\tilde{\alpha}$ are similarly defined, and $\nabla \tilde{y}(x^*)$ is the vector of derivatives $\frac{dy_j(x^*)}{dx}$ indexed by $j \in P_i$ for each $P_i \in \mathcal{S}$.

Ultimately, the constraints on α and β are

$$\begin{aligned}
x(j)-\alpha(j)-x(\bar{y}+x(\underline{y}) &\geq 0 \\
y(j)-\beta(j)-y(\bar{x})+y(\underline{x}) &\geq 0 \quad \forall j \in N \\
\alpha(S_v)+\sum_{j\in S_v\cap S_w}[x_j(\bar{y})-x_j(\underline{y})] &> x(S_v)-v(S_v) \\
\beta(S_w)+\sum_{j\in S_w\cap S_v}[y_j(\bar{x})-y_j(\underline{x})] &> y(S_w)-w(S_w,\mathcal{B}') \\
\nabla\tilde{y}(x^*)\cdot[\tilde{x}(y-\beta)-\tilde{\alpha}] &\geq \tilde{x}(y)\cdot\nabla\tilde{y}(x^*)-w(\mathcal{S},\mathcal{B}').
\end{aligned}$$

Where the third constraint must hold for every $\mathcal{S}$ such that $\mathcal{S}\cap\mathcal{P}\neq\emptyset$.

Next we must show that non-zero vectors α and β always exist. The simplest way to do so is to first recognize that this construction does not rely on which elements of the respective cores, $Co(N,v)$ and $Co(\mathcal{P},w)$ are chosen. Then we know that there exists a subset $S_v\neq P_i$ for any $P_i\in\mathcal{P}$ and a core allocation x such that $x(S_v)=v(S_v)$. Likewise, there exists a subset $S_w\neq B_k$ for any $P_i\in\mathcal{B}$ and a core allocation y such that $y(S_w)=w(S_w,\mathcal{B}')$. That is, in both cases we simply choose one of the core allocations that gives a subset of the players zero excess.

From this point, the following construction of α and β satisfies every constraint as long as we choose $S_v\cap S_w\neq S_w$. We begin with S_w. Choose one member of S_w, called j^*, that is not equal to the intersection of S_w and any member of the partition $\mathcal{P}$.

1. For all $j\neq j^*$ in S_w, $\beta(j)=0$ and $\alpha(j)=0$
2. $\beta(j^*)=\bar{\epsilon}$ and $\alpha(j^*)=0$

For all members of S_v, $\beta(j)=0$ and $\alpha(j)=x(j)$. The above constraints are reduced to:

$$\begin{aligned}
x(j)-\alpha(j) &\geq 0, \quad \textit{or} \\
y(j)-\beta(j) &\geq 0 \quad \forall j\in N \\
x(S_v) &> 0 \\
\bar{\epsilon} &> 0 \qquad\qquad (2.10)\\
\nabla\tilde{y}(x^*)\cdot[\tilde{x}(y-\beta)-\tilde{\alpha}] &\geq \tilde{x}(y)\cdot\nabla\tilde{y}(x^*)-w(\mathcal{S},\mathcal{B}').
\end{aligned}$$

The first four constraints are trivially satisfied. For the final constraint, there are $2^{|\mathcal{P}|}$ combinations of $P_i\in\mathcal{P}$. For each, the right hand side is negative or zero (only if $w(\mathcal{S},\mathcal{B}')$ and the sum of $x(j)$ for all $j\in P_i$ for all $P_i\in\mathcal{P}$ are both zero). By construction, the left hand side is positive for all values of $\beta(j^*)$ such that $\bar{\epsilon}\geq\beta(j^*)\geq 0$. We are then left with an open set

$$\left\{(\alpha,\beta(\epsilon)):\bar{\epsilon}\geq\beta(j^*)>0\right\}.$$

where the strict inequality comes from inequality 2.10 in the list of constraints above. □

Theorem 2 says that for every core of cores consisting of stable components, there is an open set of perturbed games where a core of cores consisting of unstable components exists. This also suggests that the structure $(\mathcal{P},\mathcal{B})$ is persistent. It often (i.e. in an open set) remains a part of the core of cores even when the game is perturbed.

2.2 Extensions of the Model

2.2.1 Nontransferable Payoffs with Transferable Payoffs

It is straightforward to allow any combination of transferable and nontransferable payoffs to be used on levels one and two. For nontransferable payoffs on level one we would have the traditional $V(S) \subseteq X$, where X is the set of all possible outcomes for the coalition formation game on level one. For level two we would have $W(B_k, \mathcal{B}) \subseteq Y$, where Y is the set of all possible outcomes for the coalition formation game on level two. The following definition extends the model to a game in which level one payoffs are nontransferable and level two payoffs are transferable.

Definition 2.4. A **coral game in nontransferable and transferable payoffs** is given by $\langle N, V, X, w \succsim \rangle$, where N is the number of players, $V(\cdot)$ assigns a subset of X to an element of $\mathcal{P}$, $w(\cdot,\cdot)$ assigns a real number to an element of $\mathcal{B}$, and $\succsim$ represents preferences over $X \times \mathbb{R}^+$.

Here the core of cores is defined for a game with transferable payoffs on level two and nontransferable payoffs on level one.

Definition 2.5. $(x, y, \mathcal{P}, \mathcal{B})$ is in the **core of cores** of $\langle N, V, X, w \succsim \rangle$ if $\nexists S \subset N$, with partition $\mathcal{S} = \{S_1, S_2, ...\}$, and $(x', y', \mathcal{P}', \mathcal{B}')$ such that $(x', y') \succ_S (x, y)$ where $x' \in V(S_i) \subset X$ for all $S_i \in \mathcal{S}$ and $\sum_{j \in S} y'_j = w(S, \mathcal{B})$, and $\mathcal{P}' = \{\mathcal{P} \setminus S, \mathcal{S}\}$ and $\mathcal{B}' = \{\mathcal{B} \setminus S, S\}$.

Next, the definition of the excess is adjusted to suit a game with transferable payoffs on level two and non transferable payoffs on level one.

Definition 2.6. For a coral game $\langle N, V, X, y, \succsim \rangle$ and allocations $(x, y, \mathcal{P}, \mathcal{B})$ let the **excess**, $e^*(\mathcal{S})$, for a partition, $\mathcal{S}$, of a subset of players, $S \subset N$, be the maximized value of $e(\mathcal{S})$ with respect to x'. Define $e(\mathcal{S})$ as the following:

$$e(\mathcal{S}) = [w(S, \mathcal{B}') - y(S)] - \left[\sum_{j \in S} \left[y_j(x'_j) - y_j(x_j) \right] \right] \tag{2.11}$$

and the constraints are

$$\begin{aligned} x' \in V(S_i) \subset X \text{ for each } S_i \in \mathcal{S}, \text{ and} \\ y_j(x') \geq 0 \text{ for each } j \in S, \end{aligned}$$

where $\mathcal{B}' = \{\mathcal{B} \setminus S, S\}$ and $y_j(x')$ solves $U_j(x'_j, y(x')) = \bar{U}_j = U_j(x, y)$.

The excess function is the sum of the level one and level two gains and losses accrued to $\mathcal{S}$ by deviating as a coalition of coalitions. Again, the level one payoffs are converted to level two payoffs in order to measure level one gains (or losses).

Theorem 3. *$(x, y, \mathcal{P}, \mathcal{B})$ is in the core of cores of $\langle N, V, X, w, \succsim \rangle$ if and only if $e^*(\mathcal{S}) \leq 0$ for every partition, $\mathcal{S}$, of every subset $S \subset N$.*

Proof. First, we prove that, given x^* (the allocation of $V(S_i)$ that maximizes $e(\mathcal{S})$) the condition that $e^*(\mathcal{S}) \leq 0$ is sufficient to have that there is no feasible allocation y' such that $(x^*, y') \succ_S (x, y)$. Next we will prove that, if it is sufficient given x^*, then it is sufficient for any feasible x'.

Going from x to x^* while keeping $U_j = U_j(x, y)$ for each j implies some change in level two payoffs. This is

$$\Delta y_j = y_j(y) - y_j(x') \tag{2.12}$$

where $y_j(\cdot)$ solves $U_j(x'_j, y(x')) = U_j(x, y)$

If $x^*_j \precsim_j x_j$, then agent j must be compensated in the amount of equation 2.12 in level two payoffs to keep his utility the same as when he gets (x, y). On the converse, if $x^*_j \succsim_j x_j$ then j can give up level two payoffs in the amount of equation 2.12 and be just as well off as when he gets (x, y).

Summing over $j \in S_i$ we get the net minimum gain in level two payoffs necessary to keep everyone in S as well off as before. Alternatively, this could be called the net maximum loss in level two payoffs necessary to keep everyone in S as well off as before.

Assigning level one payoffs so that each j prefers (x^*, y') to (x, y), let

$$y'_j > y_j + y_j(x) - y_j(x^*).$$

The question is whether or not y' is feasible. This requires that $\sum_j y'_j \leq w(\mathcal{S}, \mathcal{B}')$. Putting this requirement together with the level two payoffs above, we get

$$w(S, \mathcal{B}') > \sum_{j \in S} y_j + y_j(x) - y_j(x^*) = y(S) + [\sum_{j \in S} y_j(x') - y_j(x)].$$

However, this contradicts $e^*(\mathcal{S}) \leq 0$. The converse must also hold, that (x, y) is weakly preferred by all members of S to (x^*, y') whenever $e^*(\mathcal{S}) \leq 0$ holds.

To see why no other feasible (x', y') is preferred by S to (x, y), consider that x^* maximizes $e(\mathcal{S})$ by minimizing

$$-\left[\sum_{j \in S} \left[y_j(x'_j) - y_j(x_j)\right]\right],$$

and that x, $y(S)$ and $w(S, \mathcal{B}')$ are given. Therefore, if $e(\mathcal{S})$ is ever greater than zero, it must be so for $x' = x^*$.

According to the definition of the core of cores, an allocation (x, y) must be strictly preferred to (x', y') by all members of any deviating coalition, S, in any partition, $\mathcal{S}$. So if $e^*(\mathcal{S}) \leq 0$ for some $\mathcal{S}$ then there is no allocation (x^*, y') such that $\mathcal{S}$ prefers it to (x, y), and so $\mathcal{S}$ cannot viably deviate. If this holds for all $\mathcal{S}$ and $S \subset N$, then there are no viable deviators and $(x, y, \mathcal{P}, \mathcal{B})$ is in the core of cores.

The fact that the conditions are also necessary for $(x, y, \mathcal{P}, \mathcal{B})$ to be in the core of cores follows trivially from the argument above. That is, if not $e^*(\mathcal{S})$, then by the above argument, there exists feasible allocations (x^*, y'), where y'_j equals the right hand side of inequality 2.15 plus $\epsilon > 0$, such that every member of $\mathcal{S}$ strictly prefers $(x^*, y', \mathcal{P}', \mathcal{B}')$. $\square$

2.2.2 Multiple Levels

It is also straightforward to extend the coral games model to an arbitrary number of levels.

Definition 2.7. A **T-level organizational structure**, $\mathcal{P} = \{\mathcal{P}_1, ..., \mathcal{P}_T\}$, is a T-level structure where $\mathcal{P}_t = \{P_{t1}, P_{t2}, ...\}$ is a partition of $\mathcal{P}_{t-1}$ and $\mathcal{P}_1$ is a partition of N.

A function v_t assigns payoffs on level t of an organizational structure. The game at each level of the structure can be given by a characteristic function or a partition function. In either case, the payoffs can be transferable or nontransferable. For the current exposition a partition function with transferable payoffs is assumed. For each element, P_{tj}, of the partition $\mathcal{P}_t$ on each level t of the T-level organizational structure the function $v_t(P_{t,j}, \mathcal{P}_t)$ assigns a positive real number, that is the worth of that coalition: $v_t : \mathcal{P}_t \times \Omega(\mathcal{P}_{t-1}) \longrightarrow \mathbb{R}$.

Definition 2.8. A **T co-dependent organizational levels** (or **T-coral**) **game in transferable payoffs** is given by $\langle N, T, \mathbf{v}, \succsim \rangle$, where N is the number of players, $\mathbf{v} = \{v_1, ..., v_T\}$ where v_t governs the payoffs on level t and $\succsim$ represents preferences over payoffs $\mathbf{x} = (x_1, x_2, ..., x_T)$.

Before we define the stability concept for T-coral games, we must first say what constitutes a deviating coalition. This will then be a subset, $S \subset N$, that partitions into a T-level organizational structure $\mathcal{S} = \{\mathcal{S}_1, ..., \mathcal{S}_T\}$, where $\mathcal{S}_t = \{S_{t1}, S_{t2}, ...\}$ is a partition of $\mathcal{S}_{t-1}$ and $\mathcal{S}_1$ is a partition of $S \subset N$.

Definition 2.9. $(\mathbf{x}, \mathcal{P})$ is in the **T core of cores** of the T-coral game in transferable payoffs $\langle N, T, \mathbf{v}, \succsim \rangle$ if $\nexists S \subset N$, with organizational structure $\mathcal{S}$ and $(\mathbf{x}', \mathcal{P}')$ such that $\mathbf{x}' \succ_{\mathcal{S}} \mathbf{x}$ where

$$\sum_{j \in S_{ti}} x'_{t,j} = v_t(S_{ti}, \mathcal{P}'_t) \quad \forall (S_{ti}) \in \mathcal{S} \quad \text{and} \quad \mathcal{P}'_t = \{\mathcal{P}_t \setminus S, \{\mathcal{S}_t\}\} \quad \forall t.$$

For the following definition and theorem we assume that all v_t are partition functions. However, in order to allow v_t to be a characteristic function for some t, one only needs to change $v_t(S_{ti}, \mathcal{P}')$ to $v_t(S_{ti})$.

Definition 2.10. For a T-coral game in transferable payoffs, $\langle N, \mathbf{v}, \succsim \rangle$ and allocations $(\mathbf{x}, \mathcal{P})$, let the **excess**, $e^*(\mathcal{S})$, be the maximized value of $e(\mathcal{S})$ with respect to $x'_{-T} = \{x'_1, ..., x'_{T-1}\}$. Define $e(\mathcal{S})$ as the following:

$$e(\mathcal{S}) = [v_T(S, \mathcal{P}'_T) - x_T(S)] - \left[\sum_{j \in S} \int_{x_{T-1,j}}^{x'_{T-1,j}} \cdots \int_{x_{1,j}}^{x'_{1,j}} \sum_{t=1}^{T-1} \frac{dx_{T,j}(x_{t,j})}{dx_{t,j}} \prod_{t=1}^{T-1} dx_{t,j} \right] \tag{2.13}$$

and the constraints are

$$\begin{aligned} \sum_{j \in S_{ti}} x'_{t,j} &\leq v_t(S_{ti}, \mathcal{P}'_t) \\ x'_t(j) &\geq 0 \\ x_{T,j}(x'_{t,j}) &\geq 0, \quad \text{for } t < T \text{ and all } j \in S, \end{aligned}$$

where $\mathcal{P}'_t = \{\mathcal{P}_t \setminus S, \mathcal{S}_t\}$ and $\frac{dx_{T,j}(x_{t,j})}{dx_{t,j}}$ is player j's marginal rate of substitution between level T and level t payoffs when $\bar{U}_j = U_j(x_{1,j}, ..., x_{T,j})$.

Theorem 1 (revisited for T levels). $(\mathbf{x}, \mathcal{P})$ is in the core of cores of $\langle N, \mathbf{v}, \succsim \rangle$ if and only if $e^*(\mathcal{S}) \leq 0$ for every T-level structure, $\mathcal{S}$, of every subset $S \subset N$.

Proof. First, we will prove that, given x^*_{-T} (the allocation of payoffs from levels 1 through $T-1$ that maximizes $e(\mathcal{S})$) the condition that $e^*(\mathcal{S}) \leq 0$ is sufficient to have that there is no feasible allocation x'_T such that $(x^*_{-T}, x'_T) \succ_{S_i} (x_{-T}, x_T)$. Next we will prove that, if it is sufficient given x^*_{-T}, then it is sufficient for any feasible x'_{-T}.

Going from x_{-T} to x^*_{-T} along the level set $\bar{U}_j = U_j(x_{-T}, x_T)$ implies some change in $x_{T,j}(\cdot)$ in order to keep j's utility constant. By the implicit function theorem, this is

$$x_{T,j}(\cdot) = \left[\sum_{j \in S} \int_{x_{T-1,j}}^{x^*_{T-1,j}} \cdots \int_{x_{1,j}}^{x^*_{1,j}} \sum_{t=1}^{T-1} \frac{dx_{T,j}(x_{t,j})}{dx_{t,j}} \prod_{t=1}^{T-1} dx_{t,j} \right]. \tag{2.14}$$

If the constrained optimization implies $x^*_{-T,j} \succ x_{-T,j}$, then agent j must be compensated in the amount of equation 2.14 in level T payoffs to keep his utility the same as when he gets $\mathbf{x}_j$. On the converse, if $x^*_{-T,j} \prec x_{-T,j}$ then j can give up level T payoffs in the amount of equation 2.14 and be just as well off as when he gets $\mathbf{x}_j$.

Summing over $j \in S$ we get the net minimum gain in level T payoffs necessary to keep everyone in S as well off as before. Alternatively, this could be called the net maximum loss in level T payoffs necessary to keep everyone in S as well off as before.

Assigning payoffs from levels 1 to $T-1$ so that each j prefers (x^*_{-T}, x'_T) to $\mathbf{x}$, let

$$x'_{T,j} > x_{T,j} + \int_{x_{T-1,j}}^{x^*_{T-1,j}} \cdots \int_{x_{1,j}}^{x^*_{1,j}} \sum_{t=1}^{T-1} \frac{dx_{T,j}(x_{t,j})}{dx_{t,j}} \prod_{t=1}^{T-1} dx_{t,j}. \tag{2.15}$$

The question is whether or not x'_T is feasible. This requires that $\sum_j x'_{T,j} \leq v_T(S, \mathcal{P}'_T)$. Putting this requirement together with the level 1 through $T-1$ payoffs above, we get

$$\begin{aligned} v_T(S, \mathcal{P}'_T) &> \sum_{j \in S} \left[x_{T,j} + \int_{x_{T-1,j}}^{x^*_{T-1,j}} \cdots \int_{x_{1,j}}^{x^*_{1,j}} \sum_{t=1}^{T-1} \frac{dx_{T,j}(x_{t,j})}{dx_{t,j}} \prod_{t=1}^{T-1} dx_{t,j} \right] \\ &= x_T(S) + \sum_{j \in S} \int_{x_{T-1,j}}^{x^*_{T-1,j}} \cdots \int_{x_{1,j}}^{x^*_{1,j}} \sum_{t=1}^{T-1} \frac{dx_{T,j}(x_{t,j})}{dx_{t,j}} \prod_{t=1}^{T-1} dx_{t,j}. \end{aligned}$$

However, this contradicts $e^*(\mathcal{S}) \leq 0$. The converse must also hold, that $\mathbf{x}$ is weakly preferred by all members of S to (x^*_{-T}, x'_T) whenever $e^*(\mathcal{S}) \leq 0$ holds.

To see why no other feasible $\mathbf{x}'$ is preferred by $\mathcal{S}$ to $\mathbf{x}$, consider that x^*_{-T} maximizes $e(\mathcal{S})$ by minimizing

$$\sum_{j \in S} \int_{x_{T-1,j}}^{x^*_{T-1,j}} \cdots \int_{x_{1,j}}^{x^*_{1,j}} \sum_{t=1}^{T-1} \frac{dx_{T,j}(x_{t,j})}{dx_{t,j}} \prod_{t=1}^{T-1} dx_{t,j}$$

and that x_{-T}, $x_T(S)$ and $v_T(\mathcal{S}, \mathcal{P}'_T)$ are given. Therefore, if $e(\mathcal{S})$ is ever greater than zero, it must be for $x'_{-T} = x^*_{-T}$.

According to the definition of the core of cores, an allocation $\mathbf{x}$ must be strictly preferred to $\mathbf{x}'$ by all members of any deviating coalition, S, in any partition, $\mathcal{S}$. So if $e^*(\mathcal{S}) \leq 0$ for some $\mathcal{S}$ then there is no allocation (x^*_{-T}, x'_T) such that $\mathcal{S}$ prefers it to $\mathbf{x}$, and so $\mathcal{S}$ cannot viably deviate. If this holds for all $\mathcal{S}$ and $S \subset N$, then there are no viable deviators and $(\mathbf{x}, \mathcal{P})$ is in the core of cores.

The fact that the conditions are also necessary for $(\mathbf{x}, \mathcal{P})$ to be in the core of cores follows trivially from the argument above. That is, if not $e^*(\mathcal{S})$, then by the above argument, there exists feasible allocations (x^*_{-T}, x'_T), where $x'_{-T,j}$ equals the right hand side of inequality 2.15 plus $\epsilon > 0$, such that every member of $\mathcal{S}$ strictly prefers (x^*_{-T}, x'_T). □

3 Political Parties

In this section, a simple example clarifies the themes and results of coral games. As with all coral games, this example involves organization on different levels, each with separate payoffs.

On level one, politicians partition into parties. On level two, the parties play a simple majority game. These levels are co-dependent in that the outcome of the interaction between parties relies on which parties form and vice versa. Therefore, we should expect agents to make tradeoffs between payoffs on each level.

3.1 Setup

Assume that there are five politicians, $N = \{1, 2, 3, 4, 5\}$. They each have political beliefs measured in a single dimension. Let player j's beliefs be given by t_j, and order them so that $t_l > t_j$ if and only if $l > j$. For simplicity, assume that political beliefs are equally-spaced, so that $t_l - t_j = 1$ for all consecutive l and j where $l > j$.

As mentioned, politicians partition into political parties. Denote this partition as $\mathcal{P} = \{P_1, ..., P_I\}$. Let $f(P_i)$ measure the benefit to a politician of being in a party $P_i \subseteq N$, of size $|P_i|$, where $f'(\cdot) > 0$ and $f''(\cdot) < 0$. We also normalize so that $f(P_i) = 0$ if $|P_i| = 1$. Either of the functions

$$f(P_i) = \ln(|P_i|)$$

or

$$f(P_i) = 1 - \left(\frac{1}{2}\right)^{|P_i|-1}$$

will suffice.

Being in a party where the average belief differs from one's own is a drawback, so we discount $f(P_i)$ by $b|t_j - \hat{t}_i| + 1$ where $\hat{t}_i$ is the average political belief of the members in P_i and b is a positive number. Ultimately, the payoff to politician j of being in a party P_i is

$$V_j(P_i) = x(j) = \frac{f(P_i)}{b|t_j - \hat{t}_i| + 1}.$$

Once the politicians partition into parties, the parties compete for political power. We model this as a simple majority game with a small twist. The interaction between the parties

Rank	Position	Rank	Position
1	(5,0)	6	(3,1)
2	(3,0)	7	(4,1.5)
3	(4,.5)	8	(5,2)
4	(2,.5)	9	(1,0)
5	(5,1)		

Table 1: Rankings of political positions

requires that no politician may join a coalition of parties unless her entire party joins. In that way, the interaction between the parties leads to a partition of the parties, or a partition of the partition of N. Once again, denote this structure as $\mathcal{B} = \{B_1, ..., B_K\}$.

When a coalition of parties, B_k, contains three or more politicians (a majority of five), they wield political power $w(B_k) = 2 - .5(\bar{t} - \underline{t})$, where $\bar{t}$ is the belief of the right-most politician in B_k, and $\underline{t}$ is the belief of the left-most politician in the same coalition of parties. The negative second term (-.5$(\bar{t} - \underline{t})$) reflects a reduction in the power of coalitions whose members have a wide range of political beliefs. The reduction can be due to increased infighting and compromising. When they do not have a majority, a coalition of parties, B_k has zero political power ($w(B_k) = 0$). B_k is free to divide its power among its members however it chooses. An allocation, y, of $w(B_k)$ must satisfy $\sum_{j \in B_k} y(j) \leq w(B_k)$.

Summarizing, the utility function of politician j is then

$$U_j(x, y) = x(j) + \alpha y(j)$$

where $-1/\alpha$ is the constant marginal rate of substitution between payoffs on level one and level two.

3.2 Analysis

3.2.1 Stability on Each Level

To begin, assume away the interaction between the parties to look at the partitions of N that yield stability with respect to the party formation game. Then find each politician's preferences over outcomes of the form $(|S|, |t - \hat{t}_S|)$, where the first argument gives the size of the party and the second gives the politician's distance from the party's average political view. If

$$b > \frac{f(5) - f(2)}{f(2) - .5f(5)}$$

then table 1 gives rankings over positions in connected parties (i.e. $\{1, 2, 3\}$ rather than $\{1, 2, 4\}$):

The rankings follow from the fact that

$$\frac{f(5)}{b+1} > \frac{f(3)}{b+1} > \frac{f(4)}{1.5b+1}.$$

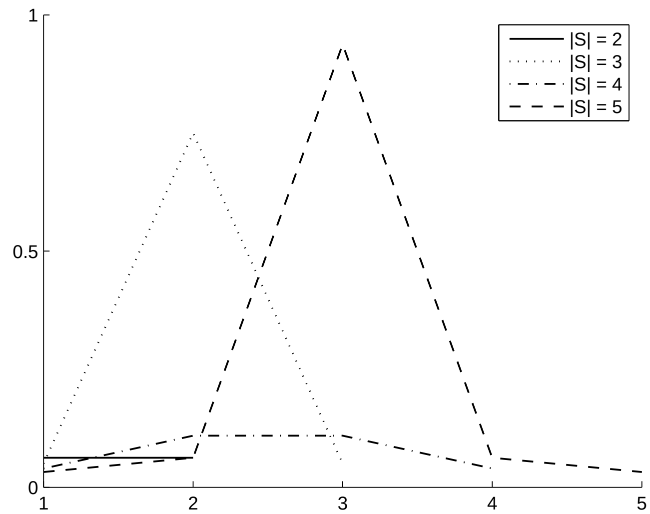

Figure 1: Party formation payoffs

The payoffs are summarized in figure 1 where each of the curves represents a different-sized party, and the height represents the payoffs to a politician in the given position.

From this it is clear that the following and their mirror images are the only stable structures of the party formation game $\langle N, V, \succsim \rangle$.

1. $\mathcal{P} = \{P_1, P_2, P_3\}$ where $P_1 = \{1, 2\}$, $P_2 = \{3\}$, and $P_3 = \{4, 5\}$
2. $\mathcal{P} = \{P_1, P_2, P_3\}$ where $P_1 = \{1, 2\}$, $P_2 = \{3, 4\}$, and $P_3 = \{5\}$
3. $\mathcal{P} = \{P_1, P_2\}$ where $P_1 = \{1, 2\}$ and $P_2 = \{3, 4, 5\}$.

For the game among coalitions, it is clear that no stable structure exists. That is, for every partition $\mathcal{B}$ and payoffs y, there is a majority subset of N, $S \neq B_k$, such that $y(S) < 2 - .5(\bar{t} - \underline{t})$.

3.2.2 Creating Instability from Stability

As just demonstrated, there does not exist a $(x, y, \mathcal{B}, \mathcal{P})$ such that $(x, \mathcal{P}) \in Co(N, V)$ and $(y, \mathcal{B}) \in Co(\mathcal{P}, w)$. So it is important to ask if the power sharing game will destabilize structures that are stable on level one. To answer this question, we use the tools from the results section above.

To begin, consider the first stable partition of the party formation game, where $\mathcal{P} = \{P_1, P_2, P_3\}$, $P_1 = \{1, 2\}$, $P_2 = \{3\}$, and $P_3 = \{4, 5\}$. For the sake of the power sharing game, consider the feasible payoffs y and the partition $\mathcal{B} = \{B_1, B_2\}$, where $B_1 = \{P_1\}$ and $B_2 = \{P_2, P_3\}$.

$S = \{1, 2, 3\}$ is a viable deviating coalition for the partition, $\mathcal{S} = \{S_1, S_2\}$ where $S_1 = \{1\}$

and $S_2 = \{2, 3\}$, when $y(3) < 1$. To see this, consider $\mathcal{S}$'s excess function,

$$\begin{aligned} e^*(\mathcal{S}) &= w(S, \mathcal{B}) - y(S) - \sum_{j \in S} [y_j(x^*) - y_j(x)] \\ &= 1 - 1 - \frac{1}{\alpha} \left[\frac{f(2)}{.5b+1} - f(1) + f(1) - \frac{f(2)}{.5b+1} \right], \end{aligned}$$

which is greater than zero if $y(3) < 1$ and zero if $y(3) = 1$.

For the case where $y(3) = 1$, the disconnected coalition $S = \{1, 2, 4\}$ is capable of deviating. First note that S improves from $y(S) = 0$ to $y'(S) = 2 - .5(3) = .5$ in the power sharing game. Now suppose they are partitioned as $\mathcal{S} = \{S_1, S_2\}$ where $S_1 = \{1, 2\}$ and $S_2 = \{4\}$. Putting this into the excess function, we get

$$e^*(\mathcal{S}) = .5 - \frac{1}{\alpha} \left[0 + 0 + \frac{f(2)}{.5b+1} - f(1)] \right].$$

Therefore, if $\alpha > 2\frac{f(2)}{.5b+1}$ then the excess is greater than zero, and $S = \{1, 2, 4\}$ can viably deviate.

In this case, by introducing a game among coalitions, we have destabilized the structure $\mathcal{P} = \{P_1, P_2, P_3\}$, where $P_1 = \{1, 2\}$, $P_2 = \{3\}$, and $P_3 = \{4, 5\}$, as corollary 3 suggests. When $y(3) = 1$, $S = \{1, 2, 4\}$ can deviate. When $y(3) < 1$, $S = \{1, 2, 3\}$ can deviate.

3.2.3 Creating Stability from Instability

As shown above, the power sharing game among coalitions can have a destabilizing effect on stable party structures. However, we would also like to know if the party formation game can have a stabilizing effect on the power sharing game. The following analysis demonstrates how this can occur.

Consider the stable party structure $\mathcal{P} = \{P_1, P_2\}$ where $P_1 = \{1, 2\}$ and $P_2 = \{3, 4, 5\}$. Let us put this together with the structure $\mathcal{B} = \{B_1, B_2\}$ where $B_1 = P_1$ and $B_2 = P_2$. We again attempt to see if $S = \{1, 2, 3\}$ is a viable deviating coalition for any partition $\mathcal{S}$. If $y(3) = 1$, then we immediately see that $\mathcal{S} = \{S\}$ cannot deviate because politician three gets the same payoff on level one and two as before. Because $w(B_1, \mathcal{B}) = 0$, therefore $y(\{1, 2\}) = 0$, and so there is no margin for making player 3 better off.

Therefore we consider the structure $\mathcal{S} = \{S_1, S_2\}$ where $S_1 = \{1\}$ and $S_2 = \{2, 3\}$. The excess is

$$\begin{aligned} e^*(\mathcal{S}) &= w(\mathcal{S}, \mathcal{B}) - y(3) - \sum_{j \in S} [y_j(x^*) - y_j(x)]] \\ &= 1 - y(3) - \frac{1}{\alpha} \left[-f(1) + \frac{f(2)}{.5b+1} - \frac{f(2)}{.5b+1} + \frac{f(3)}{b+1} \right] \end{aligned}$$

where $\frac{f(3)}{b+1} > f(1) = 0$. From this we see that the coalition $S = \{1, 2, 3\}$ cannot disrupt the structure $(\mathcal{P}, \mathcal{B})$ when $y(3) \geq 1 - \frac{1}{\alpha} \frac{f(3)}{b+1}$.

Because the connected coalition can not viably deviate, we look to see if the disconnected coalition $S = \{1, 2, 4\}$ can. Just like above, $w(\mathcal{S}, \mathcal{B}') = .5$, and when $y(3) = 1$ we have that $y(\{1, 2, 4\}) = 0$. Assume that S partitions as $\mathcal{S} = \{S_1, S_2\}$ where $S_1 = \{1, 2\}$ and $S_2 = \{4\}$. $\mathcal{S}$ has excess

$$e^*(\mathcal{S}) = .5 - 0 - \frac{1}{\alpha} [f(3)] .$$

Then, given $y(3) = 1$, $S = \{1, 2, 4\}$ can deviate if $\alpha > 2f(3)$. However, since the excess for $S = \{1, 2, 3\}$ is strictly less than zero when $y(3) = 1$, we have some slack in maintaining stability with respect to the connected coalition $S = \{1, 2, 3\}$. Specifically, as long as $y(3) \geq 1 - \frac{1}{\alpha}\frac{f(3)}{b+1}$, $(\mathcal{P}, \mathcal{B})$ is robust to deviations by partitions of $S = \{1, 2, 3\}$. Therefore, we assign $y(4) = 1 - y(3) = 1 - (1 - \frac{1}{\alpha}\frac{f(3)}{b+1})$. $\mathcal{S}$'s excess then becomes:

$$e^*(\mathcal{S}) = .5 + \frac{1}{\alpha}\frac{f(3)}{b+1} - \frac{1}{\alpha} [f(3)] .$$

This means that if $\alpha \geq 2f(3)\left[\frac{1}{b+1} + 1\right]$ then $(\mathcal{P}, \mathcal{B})$ is robust to deviations by connected and disconnected coalitions. Thus, we have created stability on level two despite the fact that the core of the power sharing game is empty.

4 Conclusion

One of the contributions of this paper is to generalize the coalitional theory to a model of multi-level structures called coral games. Cross-sections of multi-level structures are handled well by characteristic and partition functions. Therefore, these functions are the building blocks of the coral games framework.

Another contribution is to add to the understanding of complexity in game theory by analyzing the persistence of equilibria from simplified models that are embedded in larger systems. Traditionally, coalitional game theory looks at equilibria in cross sections of a multi-level structure. However, as emphasized in this paper, levels can be particularly sensitive to the influence of "nearby" games. The analysis of coral games establishes that stability can have a more complex nature than previous models have allowed. It can arise from any combination of stable and unstable components. Instability can also arise from any combination of stable and unstable components.

Political economy, which so often involves group actions, a preponderance of multi-level structures and multi-dimensional payoffs is a particularly fruitful area for applying the ideas in coral games. Interesting topics include political party interaction, coalitions of lobbyists and constitutional design.

However, applications of the coral games framework are not limited to political economy. In international trade, this type of analysis might be applied to the question of whether regional trade agreements lead to more general multilateral trade agreements [see Wei and Frankel (1996); Frankel, Stein, and Wei (1996); Yi (1996)]. It may also be useful in analyzing trends in labor organization by allowing for the structure and differential influences of labor confederations [see Chaison (1980)]. In terms of industrial organization, the coral games framework might shed light on the effect of research consortia on market structure [see Spence (1984); Katz (1986)]. And finally, the coral games framework might help us understand the types of global environmental agreements that can be reached by simultaneously accounting for international coalitions and the domestic coalitions that underlie them [see Eyckmans and Tulkens (2003)].

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