From Complex Analysis and Group Theory to Geometry and Art

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Abstract

Every college dorm displays at least one print by M.C. Escher. Many people admire Escher's work, but do not know its mathematical roots.

Escher's interest in divisions of planes goes back to his early work, but the mathematical influence in his work did not fully appear until he journeyed through the Mediterranean around 1936. Particularly, when visiting La Alhambra in Spain, he became fascinated with the order and symmetry of the tiling. He then studied mathematical papers on topics such as symmetry groups, non-Euclidean geometries, and impossible shapes, later incorporating them into his artwork.

When one looks at an Escher print, e.g. Angels and Demons, one immediately sees the complicated tiling within the circle. However, what is not necessarily realized is that the work is a representation of a hyperbolic geometric space. Though inspired by the flat tiling of the Alhambra, Escher strayed away from Euclidean geometry in many of his works, creating tiling in spherical and hyperbolic geometries. These three types of geometries make up the world we live in. On a local scale, we live on a flat surface, i.e. in Euclidean geometry, but calculating distances on Earth's surface requires spherical geometry. On a larger scale, the universe acts under the laws of hyperbolic geometry, the same as in Escher's Angels and Demons. In a similar manner, much of Escher's work is the visualization of key mathematical concepts from Complex Analysis and Group Theory.

"Mathematicians have opened the gate leading to an extensive domain." – M. C. Escher

1 Who was M. C. Escher?

When studying mathematics, new ideas are presented in an abstract, highly conceptualized form. It is easy to get lost in these intangibles and forget just how beautiful math can be. But behind every mathematical theorem is a picture - a means of visualizing these otherwise abstract ideas. A wonderful way to visualize mathematics is through the artwork of Dutch artist, M. C. Escher.

Despite never having any formal, mathematical training, Escher became known for the strong mathematical influences in his artwork. He was inspired by his travels around Europe and forays into mathematical papers to create his own artistic renditions of mathematical concepts on topics such as non-Euclidean geometries, impossible shapes, and symmetry groups. He began his career studying architecture, but, after a poor academic performance, switched to painting and sculpture. From then on, his artwork took the form of woodcuts, lithographs, and mezzotints.

Escher strayed from his early works of landscapes and nature and ventured into the world of mathematical art after a visit to the Spanish palace, La Alhambra. Originally constructed as a fortress, but later turned into a royal palace in the 14th century, La Alhambra remains a popular tourist destination, exhibiting some of Spain's most impressive Moorish architecture and art. In tandem with Moorish tradition, the walls and ceilings of the palace are sculpted heavily with intricate tiling, or interlocking, repetitive patterns laid out in geometric grid.

Figure 1: Tiling at La Alhambra

After seeing the patterns at La Alhambra, Escher became obsessed with the regular division of the plane. Returning home, he attempted his own sketches and showed them to his brother, Berend, who in turn introduced him to a paper on plane symmetry groups by the mathematician, Pólya. Despite not understanding many of the theoretical concepts, Escher was able to teach himself the principles behind each of the 17 groups of symmetry for a 2-dimensional plane. The figure below presents some excellent examples of Escher's tilings. The complicated morphing within the tilings demonstrate the geometric bases for each pattern ("M. C. Escher").

Figure 2: Escher's woodcut *Metamorphosis II* (1940)

In 1954, Escher met the British-born Canadian mathematician, Harold Scott MacDonald Coxeter, who was considered one of the greatest geometers of the 20th century. They became life-long friends, and Escher read one of his articles on hyperbolic tessellations, or tiling on a hyperbolic disk, the concept of which inspired Escher's *Circle Limit* series. A couple decades after Escher's death, Coxeter, in turn, studied his etchings, and came to the conclusion that Escher got the tessellations right, down to the millimeter (O'Connor).

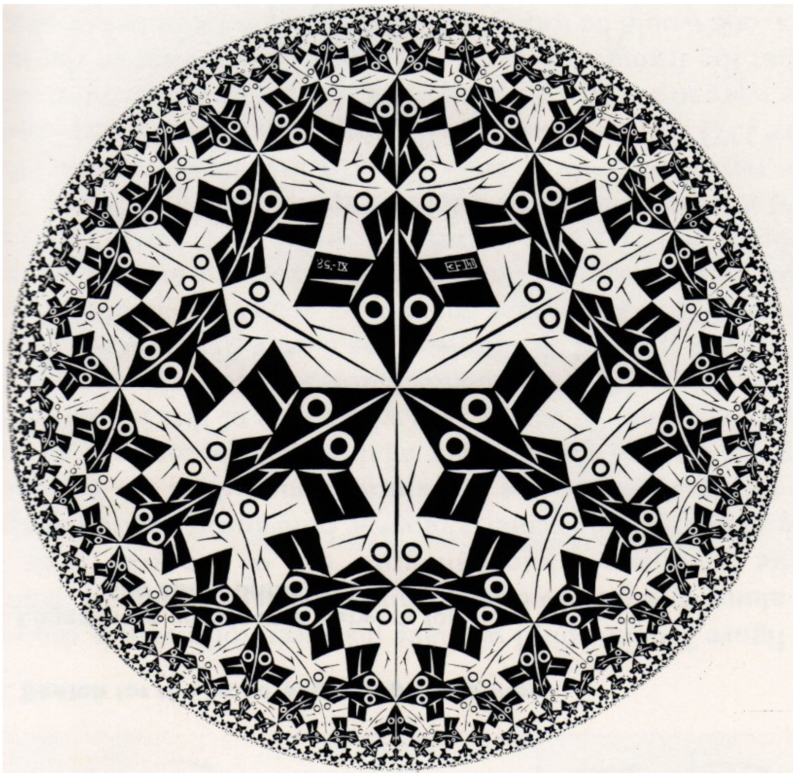

Figure 3: Escher's woodcut *Circle Limit I* (1958)

Escher's own mathematical pursuits culminated in a paper, titled *Regular Divisions of the Plane* (1958), in which he described the systematic buildup of mathematical designs in his artwork.

At this point, Escher grew tired of the regular divisions of the plane and he turned to more unusual perspectives, exploring different geometries, as in *Circle Limit*, as well as depicting impossible shapes that cannot be realized in a 3-dimensional world. One of the shapes Escher became fascinated with is the Penrose triangle, popularized by the mathematician Roger Penrose. The triangle has the illusion of being solid on a flat surface, but it cannot be built in reality. In his lithograph, *Waterfall*, Escher overlays two impossible triangles to create a self-feeding waterfall that ends up two stories higher than it began ("M. C. Escher"). Escher began his sketch of the lithograph with the triangular form of a Moebius band, which is a non-orientable surface in which one can circle around the surface twice and end up back in the same exact spot.

Figure 4: Escher's lithograph *Waterfall* (1961)

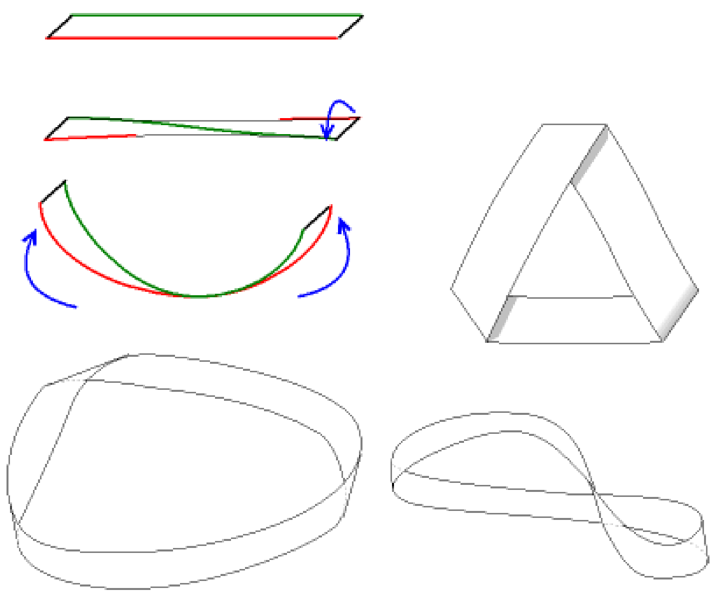

Figure 5: Making a Moebius Band

However, despite his change in focus, Escher did not completely lose his roots in drawing insects and using themes from nature, often overlaying his geometric grids with designs with insects or other types of animals. He incorporated these images into his multitude of geometric tilings. As seen in *Metamorphosis II* and *Circle Limit I*, Escher incorporated a lot of movement and patterns into this work. Such motions in Escher's artwork can be explained through concepts from Complex Analysis and Group Theory.

2 Complex and Linear Mappings

Complex, or linear, mappings help to explain the movement in the tiling in Escher's artwork. A linear mapping is a transformation of vectors from the domain to the range, that preserves addition and scalar multiplication.

So for a function, $f : S \to T$, if $x_1, x_2 \in S$, then

$$f(x_1 + x_2) = f(x_1) + f(x_2)\, and$$

$$f(ax_1) = af(x_2).$$

A mapping can also be thought of as a motion, which comes from the representation of a complex number as a 2 x 2 matrix. Linear mappings have three types of transformations: dilation, rotation, and translation.

Complex numbers are the extension of the real number system and can be viewed as points of the coordinate plane. A complex number, z, can be expressed in the form of $z = a + bi$, where a is the real part and y is the imaginary part, and where $i^2 = -1$. Here, a and b are real numbers and can also be expressed as an ordered pair, (a, b). Since complex numbers are two-dimensional, they can be represented by 2 x 2 matrices.

The simplest way to understand how a complex number can be written as a matrix, is to use Euler's formula in combination with polar form.

If $z = a + ib$, then

$r = |z| = \sqrt{a^2 + b^2}$, which is the modulus, or absolute value, of z, and
$arg(z) = \theta = tan^{-1}(\frac{b}{a})$, which is the argument, or the angle of r with the positive real axis.

So in polar, $z = rcos(\theta) + irsin(\theta)$.
But, Euler's formula says that $e^{ix} = cos(x) + isin(x)$.

So,

$$e^{ix} = cos(x) + isin(x)$$

$$\iff re^{ix} = r(cos(x) + isin(x)) = rcos(x) + irsin(x).$$

Hence, $z = re^{i\theta}$.

Something to note, is that $|e^{i\theta}| = \sqrt{cos^2(\theta) + sin^2(\theta)} = 1$, indicating that $e^{i\theta}$lives on the unit circle. When multiplying by r, $|re^{i\theta}| = r$, making r the radius of the circle.

To finish, note that in the complex plane, the number 1 can be expressed by the point $(1, 0)$, or in matrix form, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Likewise, the point i can be expressed by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers. Then,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \, and$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So multiplication by the number 1, takes number to i and multiplication by the number i, takes it to -1. Multiplication by i becomes a rotation by $90°$. Solving,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

we get

$$z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

We can then use polar, where $a = rcos(\theta)$ and $b = rsin(\theta)$, to obtain

$$z = \begin{bmatrix} rcos(\theta) & -rsin(\theta) \\ rsin(\theta) & rcos(\theta) \end{bmatrix} = r \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix},$$

where $\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$ is a rotation matrix.

Also, by DeMoivre's Theorem, $z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n cos(n\theta) + ir^n sin(n\theta)$. So z^n is a rotation of $\mathbb{C}$ upon itself n-times (Marsden and Hoffman 16).

3 Geometry

In order to understand Escher's work, we need to look at different types of geometries by the set of motions that preserve the geometries. There are three types of geometries in the world and we experience all three of these types on different scales.

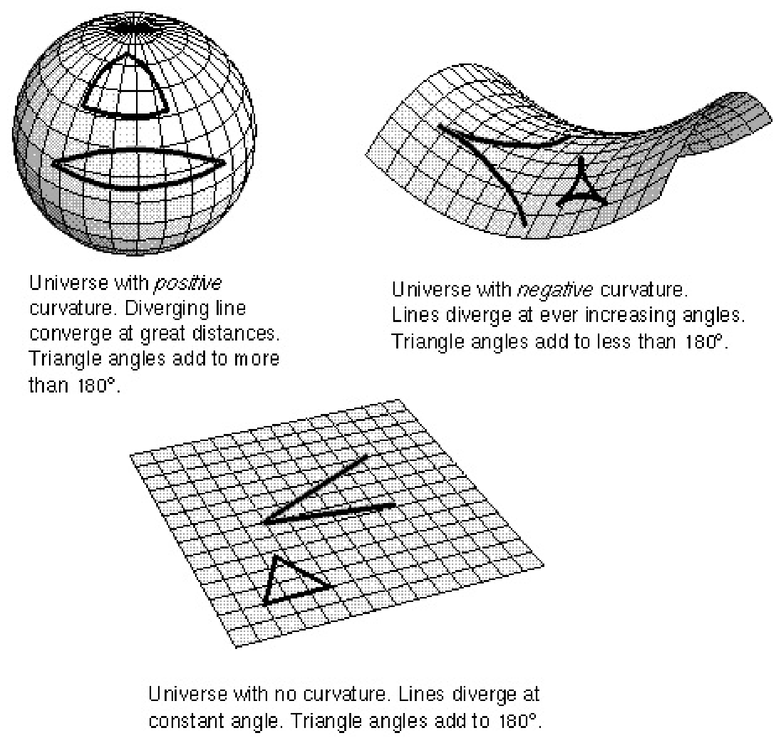

Figure 6: Curvature and Geometry

On a daily basis, we experience flat, or Euclidean geometry. In this geometry, triangles have at most 180°, and all of the trigonometric properties we are used to, hold. When we calculate walking distances or driving distances, we use Euclidean geometry.

Since the Earth is a sphere, on a larger scale we experience spherical geometry. On the Earth's surface, triangles can have anywhere between 180° and 360°. In spherical geometry, lines are mapped to circles since any line drawn on the Earth's surface can be thought of as an equator, or a circle. On such a surface, parallel lines always eventually cross. A common way spherical geometry is used is in the calculation of flight paths.

The third type of geometry, and perhaps the least intuitive, is hyperbolic geometry. We experience this geometry without often realizing it. Although the shape of the universe is still an open question, some popular theories suggest that the universe has a hyperbolic nature. Due to gravitational pulls, things get pulled inwards.

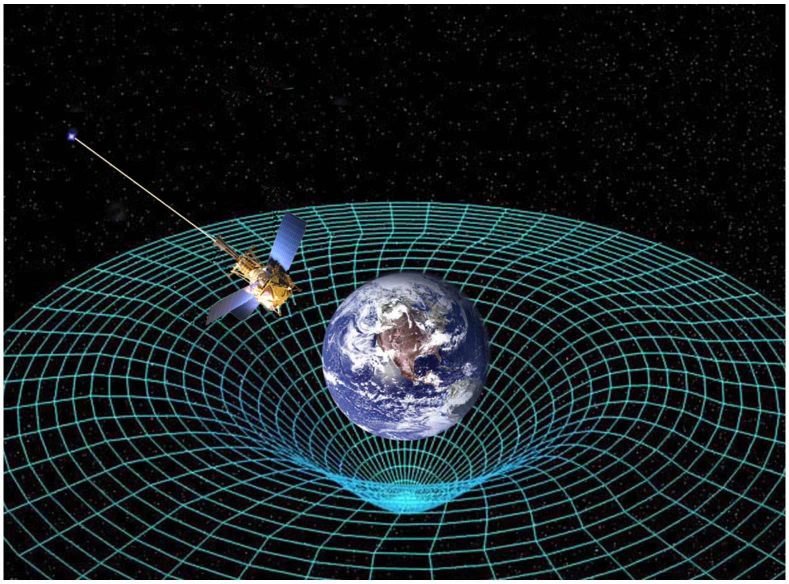

Figure 7: Curvatures of Space

In hyperbolic geometry, the angles of triangles add up to less than 180°. An ideal hyperbolic triangle, which has all three vertices lying on the circle at infinity, has interior angles of zero, and therefore has 0°.

Another example of hyperbolic geometry, is the internet. Since the internet was grown organically, with most of the routes passing through the core. It can be observed that in order to connect any three nodes, the routes must bend inwards.

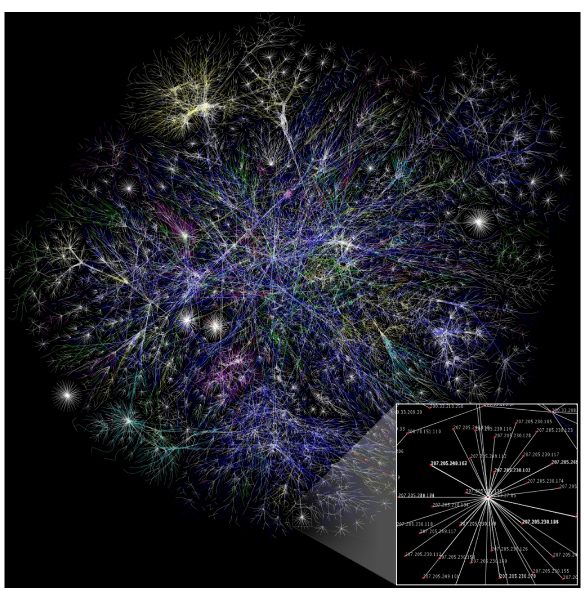

Figure 8: Internet Map

Locally, all geometries look flat. Just like on the Earth's surface, we perceive ourselves living on a flat surface, both spherical and hyperbolic geometries can be locally approximated using linear approximations.

The following theorem proves that these three geometries are the only three geometries.

Theorem. *Uniformization Theorem [Klein (1883), Koebe (1900's), Poincare (1907)]:*
(i.) Every surface admits a Riemannian metric of constant Gaussian curvature.
(ii.) Every simply connected Riemann surface is conformally equivalent to one of the following:

$$\mathbb{C}(K=0);\ \hat{\mathbb{C}}(K=1);\ \mathbb{D}(K=-1).$$

The Uniformization Theorem states that any simply-connected Riemann surface is conformally equivalent to, or can be mapped to, one of three surfaces. A Riemann surface is a generalization of the complex plane, which has the following property: any small disk on the surface can be conformally mapped to a corresponding disk in the complex plane. The surfaces include the complex plane ($\mathbb{C}$), the Riemann sphere ($\hat{\mathbb{C}}$), and the hyperbolic unit disk ($\mathbb{D}$). The Uniformization Theorem says that these are the only three conformally distinct Riemann surfaces.

If the curvature equals 0, then the surface can be conformally mapped to the complex plane. It would therefore follow the geometry of flat surfaces, or Euclidean geometry. If the curvature equals $+1$, then the surface can be conformally mapped to the sphere, and would therefore fall under spherical geometry. If the curvature equals -1, then the surface can be conformally mapped to the hyperbolic plane, and would then follow the laws of hyperbolic geometry. As a result, the uniformization theorem shows that the world exhibits three, and only three, types of geometries (Casey).

4 Groups

Before discussing the motions of each geometry, we must introduce groups.

A Group is a set, $\mathbb{G}$, and an operation *, denoted

$$\langle \mathbb{G}, * \rangle$$

such that:

- For all $g, h \in \mathbb{G}$, $g * h \in \mathbb{G}$, i.e. the set is closed.
- For all $g, h, k \in \mathbb{G}$, $(g * h) * k = g * (h * k)$, i.e. associativity holds.
- There exists $e \in \mathbb{G}$ such that for all $g \in \mathbb{G}$, $g * e = g$, i.e. there is an identity element in the group.
- For all $g \in \mathbb{G}$, there exists $g^{-1} \in \mathbb{G}$ such that $g * g^{-1} = e$, i.e. every element has an inverse.

The operation * can also be thought of as a motion.

5 Groups of Motions in Each Geometry

Each type of geometry can be represented by its group of motions, with composition as the operator:

- Euclidean Geometry

$$\langle \{e^{i\theta}z+\alpha\}, \circ\rangle$$

- Spherical Geometry

$$\left\langle \left\{ \frac{\alpha z-\beta}{-\bar{\beta}z+\bar{\alpha}} \right\}, \circ \right\rangle,$$

where $|\alpha|^2+|\beta|^2=1$.

- Hyperbolic Geometry

$$\left\langle \left\{ e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z} \right\}, \circ \right\rangle,$$

where $|\alpha|<1$

As mentioned previously, in Euclidean geometry, the mappings are called linear transformations, which comprise of three forms of motions, dilation, rotation, or translation.

In spherical geometry, the mappings are called Moebius Transformations, which are bi-linear mappings. In addition to the three types of motion from linear transformations, Moebius Transformations add the extra action of inversion. An example of an inversion is $\frac{1}{z}$, which turns $\mathbb{C}$, the complex plane, inside out. Another, $\left(\frac{1}{z}\right)^2$ turns $\mathbb{C}$ inside out and then folds $\mathbb{C}$ onto itself. Spherical mappings now do the four following - dilation, rotation, translation, and inversion. Geometrically, circles are mapped to circles, but now lines are also mapped to circles, because, as mentioned previously, lines can be thought of as equators.

In hyperbolic geometry, the mappings are called Moebius-Blaschke Transformations, and these describe hyperbolic motion (Casey).

6 Euclidean Geometry in Escher's Work

Escher began his mathematical art after observing Moorish tilings. Accordingly, he began his creating own tilings on a flat plane, using linear transformations from Euclidean geometry to rotate, translate, and dilate his patterns. Escher's tilings can be categorized into the following 17 symmetry groups, with each symmetry group describing a different set of motions. The symmetry groups are as follow:

Symmetry Group	Lattice	Rotation Order	Reflection Axis	Generating Region
p1	Parallelogram	1	none	1 unit
p2	Parallelogram	2	none	$\frac{1}{2}$ unit
pm	Rectangular	1	parallel	$\frac{1}{2}$ unit
pg	Rectangular	1	none	$\frac{1}{2}$ unit
cm	Rhombic	1	parallel	$\frac{1}{2}$ unit
pmm	Rectangular	2	90°	$\frac{1}{4}$ unit
pmg	Rectangular	2	parallel	$\frac{1}{4}$ unit
pgg	Rectangular	2	none	$\frac{1}{4}$ unit
cmm	Rhombic	2	90°	$\frac{1}{4}$ unit
p4	Square	4	none	$\frac{1}{4}$ unit
p4m	Square	4	45°	$\frac{1}{8}$ unit
p4g	Square	4	90°	$\frac{1}{8}$ unit
p3	Hexagonal	3	none	$\frac{1}{3}$ unit
p3m1	Hexagonal	3	60°	$\frac{1}{6}$ unit
p31m	Hexagonal	3	30°	$\frac{1}{6}$ unit
p6	Hexagonal	6	none	$\frac{1}{6}$ unit
p6m	Hexagonal	6	30°	$\frac{1}{12}$ unit

Figure 9: 17 Groups of Symmetry (Gallian 476)

Here, p1 is the most basic symmetry group, comprising of only translations and no rotational or reflective symmetry. An example of a p1 symmetry group can be seen in Escher's *Sky and Water*, where the birds and the fish are only shifted, but not rotated and reflected.

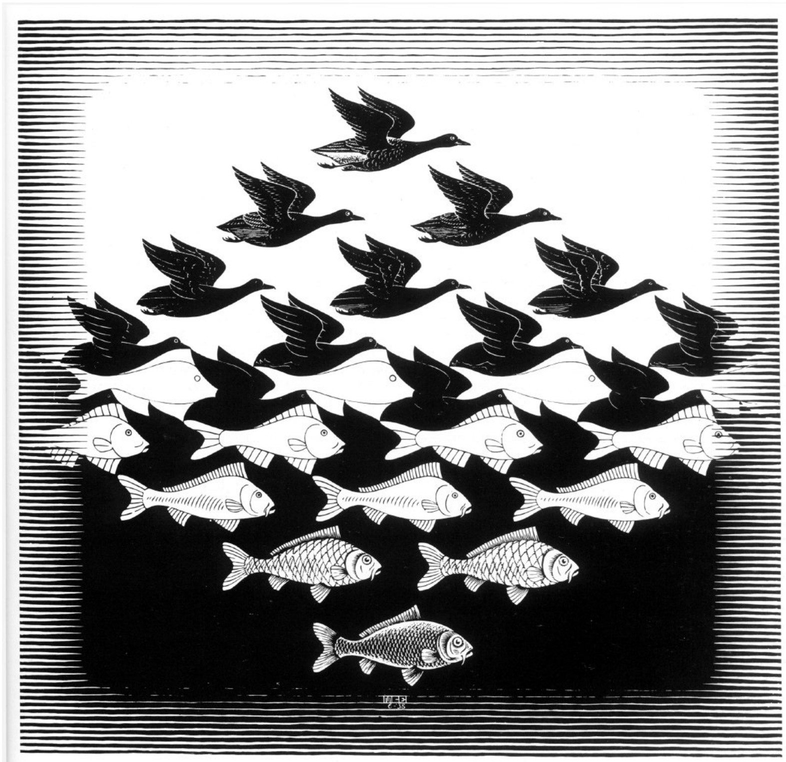

Figure 10: Escher's woodcut *Sky and Water I* (1938)

Even Escher's famous *Day and Night* exemplifies these mappings. Disregarding the areas where the birds morph into other aspects of the artwork, and only focusing on the tiling of the birds, the pattern is once again characteristic of the p1 symmetry group. Each combined pair of black and white birds is only shifted along the page and neither rotated or reflected.

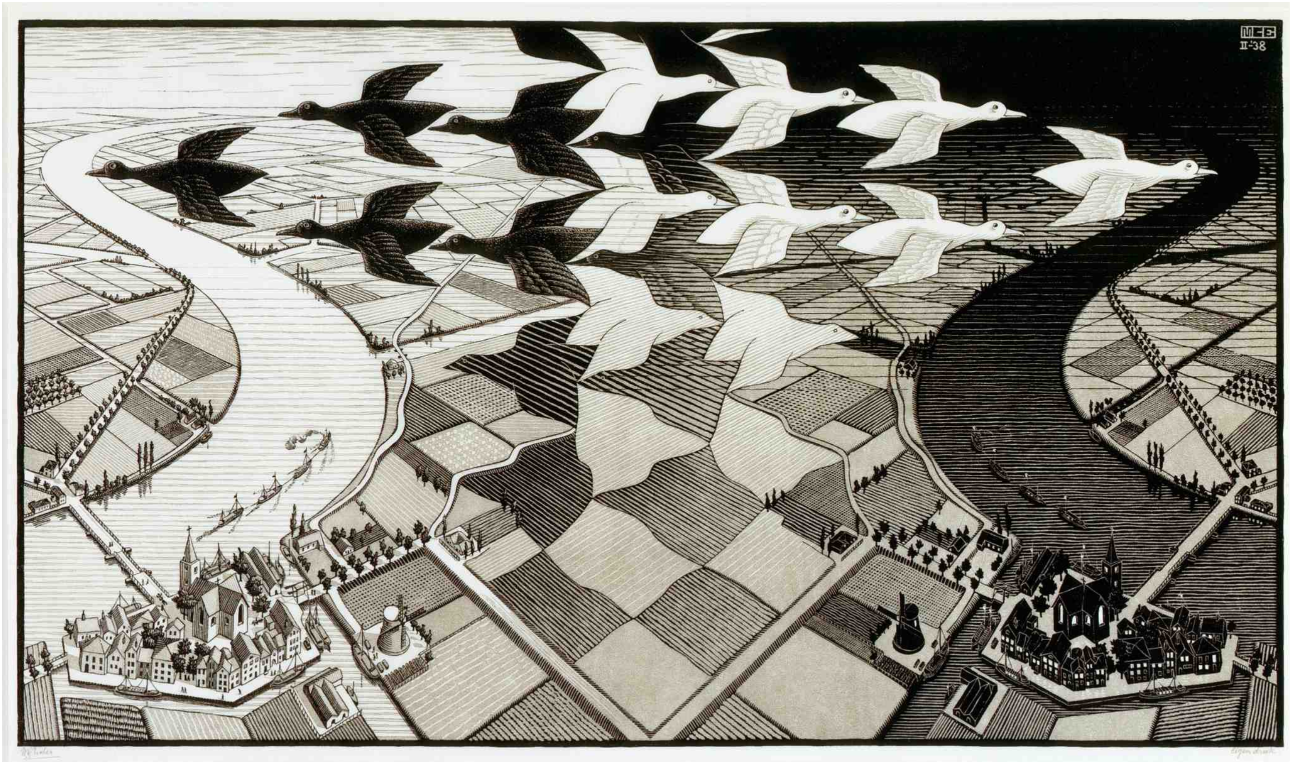

Figure 11: Escher's woodcut *Day and Night* (1938)

Escher's *Sun and Moon* is characteristic of a different symmetry group, p31m, due to the 120° rotations. Each figure can be rotated 120° around certain points to bring it back to its original spot. Each bird can be treated as triangle that is reflected over its three edges. Since this piece has the 120° rotation, reflection, and all threefold centers lie on the reflection axes, it can be classified into the p31m group. However, Escher did make slight modifications to the beak and wing structures of the birds as they go out from the center.

Figure 12: Escher's woodcut *Sun and Moon* (1948)

7 Spherical Geometry in Escher's Work

While not as numerous as his flat or hyperbolic works of art, Escher also experimented with spherical mappings. The piece, *Hand with Reflecting Sphere* is a great example of spherical geometry. The reflection within the sphere follows curvature of a Moebius mapping. Since Moebius transformations are distance-preserving, the distances of all of the objects in the room are preserved in the reflecting sphere. Moebius transforms form the group of isometries, or the set of all distance-preserving maps, on the sphere.

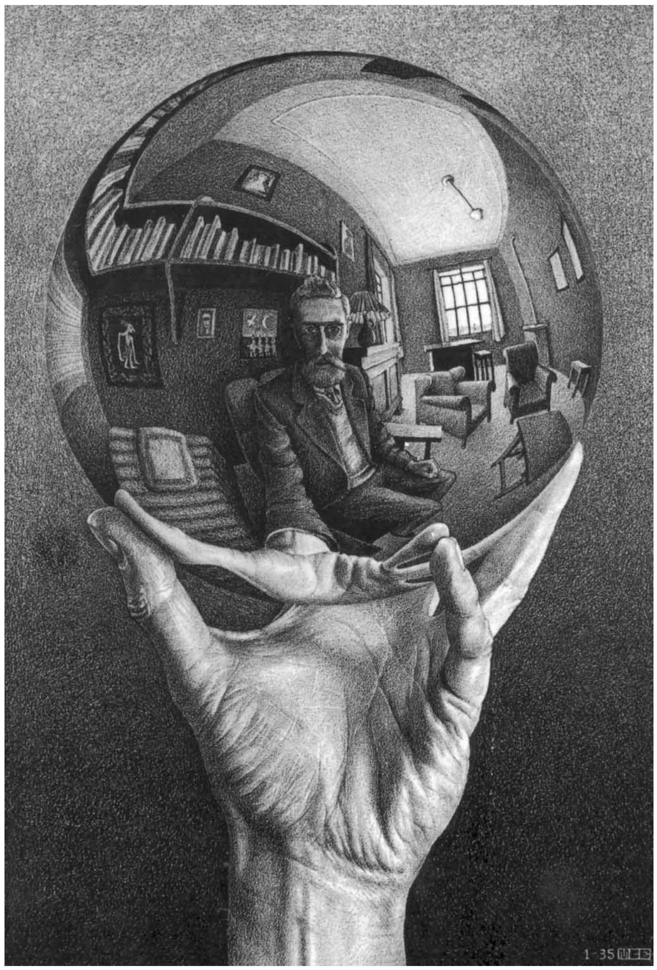

Figure 13: Escher's lithograph *Hand with Reflecting Sphere* (1935)

Using a bit of Complex Analysis, we can show that the sphere in the artwork preserves the distances of the objects in the room by showing that the Moebius mappings preserve length. To do this, we need to consider

$$\mathscr{L}(\Gamma) = \oint_{\Gamma} \tfrac{|dz|}{1+|z|^2},$$

where $\mathscr{L}(\Gamma)$ is the length of the path Γ in the unit circle. As mentioned above, $M_{\alpha,\beta}(z) = \frac{\alpha z+\beta}{-\bar{\beta} z+\bar{\alpha}}$ is the Moebius transformation.

Claim. We want to show that $\mathscr{L}(M_{\alpha,\beta}(\Gamma)) = \mathscr{L}(\Gamma)$, i.e. that the length of Γ with the Moebius transformation acting upon it is equal to the original length of Γ.

Proof. $\mathscr{L}(M_{\alpha,\beta}(\Gamma)) = \oint_{M_{\alpha,\beta}(\Gamma)} \frac{|dz|}{1-+|z|^2} = \oint_\Gamma \frac{|M'_{\alpha,\beta}(z)||dz|}{1+|M_{\alpha,\beta}(z)|^2}$. Since $M_{\alpha,\beta}(z) = \frac{\alpha z+\beta}{-\bar{\beta}z+\bar{\alpha}}$, $M'_{\alpha,\beta}(z) = \frac{\alpha(-\bar{\beta}z+\bar{\alpha})+\bar{\beta}(\alpha z+\beta)}{(-\bar{\beta}z+\bar{\alpha})^2} = \frac{-\alpha\bar{\beta}z+\alpha\bar{\alpha}+\alpha\bar{\beta}z+\beta\bar{\beta}}{(-\bar{\beta}z+\bar{\alpha})^2} = \frac{|\alpha|^2+|\beta|^2}{|-\bar{\beta}z+\bar{\alpha}|^2}$, since $\alpha\bar{\alpha} = (a+ib)(c+id) = a^2+b^2 = |\alpha|^2$ by definition. Then $\mathscr{L}(M_{\alpha,\beta}(\Gamma)) = \oint_\Gamma \frac{\frac{|\alpha|^2+|\beta|^2}{|-\bar{\beta}z+\bar{\alpha}|^2}}{1+|\frac{\alpha z+\beta}{-\bar{\beta}z+\bar{\alpha}}|^2}|dz| = \oint_\Gamma \frac{|\alpha|^2+|\beta|^2}{|-\bar{\beta}z+\bar{\alpha}|^2+|\alpha z+\beta|^2}|dz|$
$= \oint_\Gamma \frac{|\alpha|^2+|\beta|^2}{|\bar{\beta}|^2|z|^2-2|\bar{\alpha}\bar{\beta}z|+|\bar{\alpha}|^2+|\alpha|^2|z|^2+2|\alpha\beta z|+|\beta|^2}|dz|$. Since $|\alpha| = |\bar{\alpha}|$ and $|\beta| = |\bar{\beta}|$, this equals,
$\oint_\Gamma \frac{|\alpha|^2+|\beta|^2}{|\beta|^2|z|^2+|\alpha|^2|z|^2+|\alpha|^2+|\beta|^2}|dz| = \oint_\Gamma \frac{|\alpha|^2+|\beta|^2}{(|\alpha|^2+|\beta|^2)(1+|z|^2)}|dz| = \oint_\Gamma \frac{|dz|}{1+|z|^2} = \mathscr{L}(\Gamma)$. □

Since the length of any path with the Moebius transformations acting on it is equal to the original length of the path, Moebius mappings are length-preserving.

8 Hyperbolic Geometry in Escher's Work

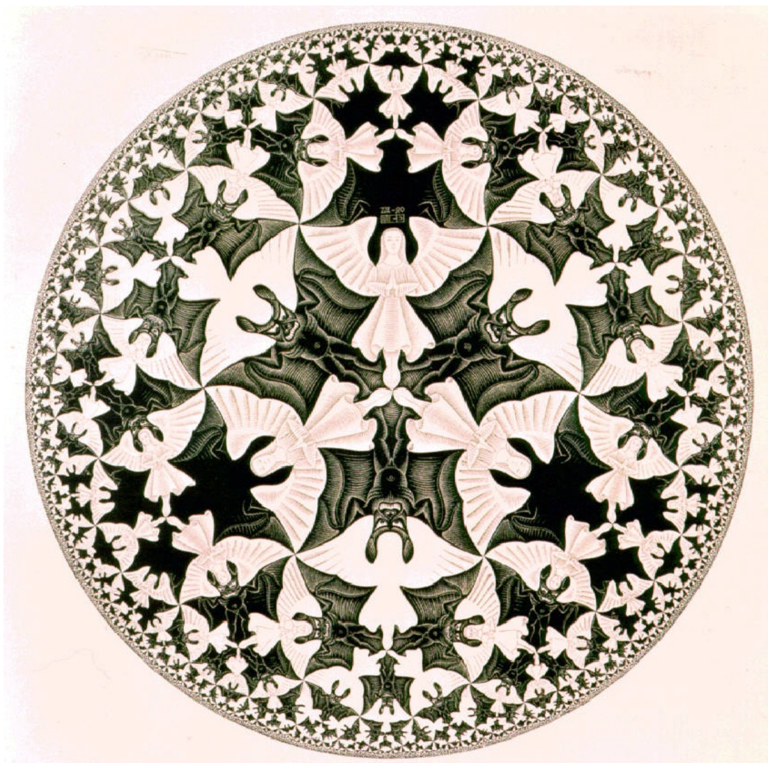

Figure 14: Escher's woodcut *Angels and Demons* (1960)

One of the most famous Escher pieces, *Angels and Demons*, is a great example of hyperbolic mappings. The piece is a representation of hyperbolic geometry on a hyperbolic disk. Each angel is mapped onto the other angels through rotations and translations on the hyperbolic plane. Likewise, rotations and translations are used to map each demon onto the other demons.

As with the Moebius mapping, we can show that the Moebius-Blaschke motions preserve hyperbolic length. Even though when we view piece on a flat surface, the angels and demons on the outer edges of the disk appear smaller than the ones in the center, in the hyperbolic plane, each angel has the same exact area. Similarly, every demon has equal area.

In order to show that the hyperbolic motions preserve length, we need to consider

$$\mathscr{L}(\Gamma) = \oint_\Gamma \frac{|dz|}{1-|z|^2},$$

where $\mathscr{L}(\Gamma)$ is the hyperbolic length of the path Γ in the unit disk. As mentioned above, $\Phi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ is the Moebius-Blaschke transformation.

Claim. We want to show that $\mathscr{L}(\Phi_\alpha(\Gamma)) = \mathscr{L}(\Gamma)$, i.e. that the length of Γ with the Moebius-Blaschke transformation acting upon it is equal to the original length of Γ.

Proof. $\mathscr{L}(\Phi_\alpha(\Gamma)) = \oint_{\Phi_\alpha(\Gamma)} \frac{|dz|}{1-|z|^2} = \oint_\Gamma \frac{|\Phi'_\alpha(z)||dz|}{1-|\Phi_\alpha(z)|^2}$. Since $\Phi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$, $\Phi'_\alpha(z) = \frac{(1-\bar{\alpha}z)+\bar{\alpha}(z-\alpha)}{(1-\bar{\alpha}z)^2} = \frac{1-\bar{\alpha}z+\bar{\alpha}z-\alpha\bar{\alpha}}{(1-\bar{\alpha}z)^2} = \frac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2}$, since $\alpha\bar{\alpha} = (a+ib)(c+id) = a^2+b^2 = |\alpha|^2$ by definition.

Then $\mathscr{L}(\Phi_\alpha(\Gamma)) = \oint_\Gamma \frac{|\Phi'_\alpha(z)||dz|}{1-|\Phi_\alpha(z)|^2} = \oint_\Gamma \frac{\frac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2}}{1-|\frac{z-\alpha}{1-\bar{\alpha}z}|^2}|dz| = \oint_\Gamma \frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2-|z-\alpha|^2}|dz|$ $= \oint_\Gamma \frac{1-|\alpha|^2}{1-2|\bar{\alpha}z|+|\alpha|^2|z|^2-|z|^2+2|\alpha z|-|\alpha|^2}|dz|$. Since $|\alpha| = |\bar{\alpha}|$, this equals, $\oint_\Gamma \frac{1-|\alpha|^2}{1+|\alpha|^2|z|^2-|z|^2-|\alpha|^2}|dz| = \oint_\Gamma \frac{1-|\alpha|^2}{(1-|\alpha|^2)(1-|z|^2)}|dz| = \oint_\Gamma \frac{|dz|}{1-|z|^2} = \mathscr{L}(\Gamma)$. □

Once again the length of the path with the Moebius-Blaschke transformation acting on it is equal to the original length of the path.

The same hyperbolic mapping can be seen in Escher's *Circle Limit III*. As before, the area of each triangle and of each fish is preserved. In this piece, it is also easy to see the inward bending curvature of the triangles in hyperbolic geometry.

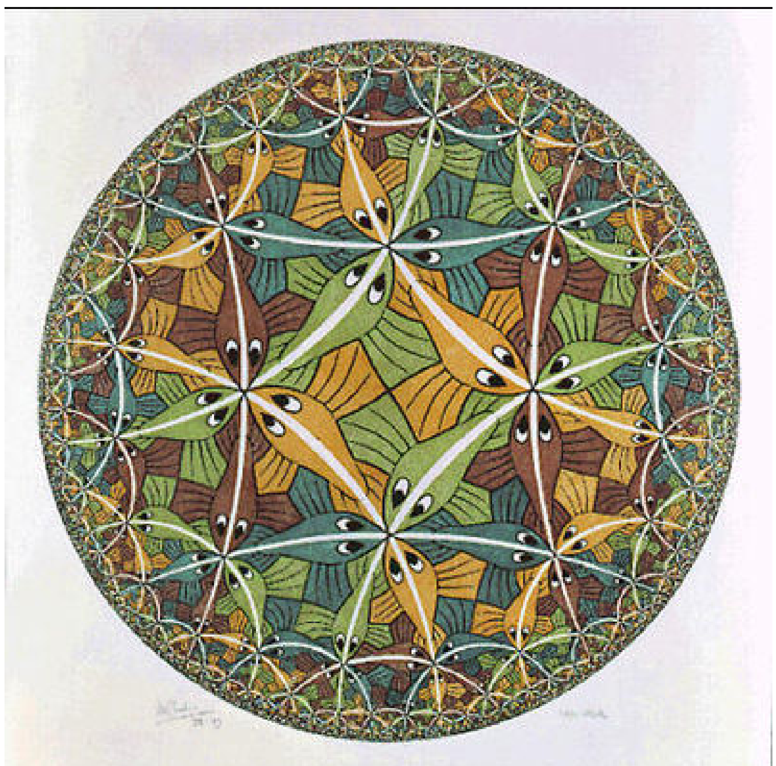

Figure 15: Escher's woodcut *Circle Limit III* (1959)

An interesting thing to note, is that since the disk is representing the hyperbolic plane, the edge of the piece is the circle at infinity.

9 Conclusion

These are only a small sampling of Escher's works of art, but they are emblematic of his approach. Even without formal mathematical training, Escher understood that math could be applied in unexpected ways. The tools from Complex Analysis and Group Theory help formalize the principles that Escher grasped intuitively. They explain the systematic approach that Escher took in creating movement in his art. In addition to being great works of art, his artwork forms a medium of expression for these abstract ideas. Next time an Escher print catches your eye in a college dorm, you can stop and reflect on not only the artistic, but also the mathematical complexity inherent in the work.

References

[1] Casey, Stephen D. "Complex Analysis." American University, Washington, DC. Lecture Notes.

[2] Curvature and Geometry. Digital image. *UF Astronomy*. University of Florida, n.d. Web.

[3] Escher, M. C., and J. L. Locher. *The World of M. C. Escher*. New York: H.N. Abrams, 1971. Print.

[4] Escher, M. C. *Angels and Demons*. 1960. *The Official M. C. Escher Website*. M. C. Escher Foundation. Web.

[5] Escher, M. C. *Circle Limit I*. 1958. *The Official M. C. Escher Website*. M. C. Escher Foundation. Web.

[6] Escher, M. C. *Circle Limit III*. 1959. *The Official M. C. Escher Website*. M. C. Escher Foundation. Web.

[7] Escher, M. C. *Day and Night*. 1938. *The Official M. C. Escher Website*. M. C. Escher Foundation. Web.

[8] Escher, M. C. *Hand with Reflecting Sphere*. 1935. *The Official M. C. Escher Website*. M. C. Escher Foundation. Web.

[9] Escher, M. C. *Metamorphosis II*. 1940. *Metamorphosis II*. Wikipedia. Web.

[10] Escher, M. C. *Sky and Water I*. 1938. *The Official M. C. Escher Website*. M. C. Escher Foundation. Web.

[11] Escher, M. C. *Sun and Moon*. 1948. *The Official M. C. Escher Website*. M. C. Escher Foundation. Web.

[12] Escher, M. C. *Waterfall*. 1961. *The Official M. C. Escher Website*. M. C. Escher Foundation. Web.

[13] Gallian, Joseph A. *Contemporary Abstract Algebra*. Belmont, CA: Brooks/Cole, Cengage Learning, 2010. Print.

[14] Internet Map. Digital image. N.p., n.d. Web.

[15] Marsden, Jerrold E., and Michael J. Hoffman. *Basic Complex Analysis*. New York: W.H. Freeman, 1999. Print.

[16] Making a Moebius Band. Digital Image. *The Moebius Band and Other Surfaces*. Saint Louis University, n.d. Web.

[17] “M. C. Escher." *Wikipedia*. Wikimedia Foundation, 04 Mar. 2013. Web. 07 Apr. 2013.

[18] O'Connor, J. J., and E. F. Robertson. "Maurits Cornelius Escher." University of St. Andrews, May 2000. Web. 03 May 2013.

[19] Possible Curvature of Space. Digital Image. *A Curious Mind*. Space Telescope Science Institute, 19 June 2012. Web.

[20] Tiling at La Alhambra. Digital image. *This Week's Finds in Mathematical Physics*. University of California, Riverside, n.d. Web.