

VaR versus Expected Shortfall and Expected Value Theory

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A. Risk management in the twenty-first century

A lesson learned from the recent financial crisis is how important risk management is in today's ever more globalized world. Financial markets in this day and age do not work in isolation. The idea of a contagion spreading from a sovereign crisis in Europe to the U.S. is not out of the question anymore. The intertwined nature in which markets work today is the reason why risk management has taken a central stage in today's financial systems.

Firms that are able to accurately assess and manage different types of risk ultimately increase their profitability over time. The definition of risk, though, is too broad. Risk can be classified as one of several forms. Following are some of the types of risks that almost every firm today directly or indirectly faces.

1. Operational risk – is a broad concept that encompasses all sorts of monetary losses that may result from “inadequate or failed internal processes, people, and systems or from external events” (Federal Reserve Bank of San Francisco, 2002).
2. Credit risk – is the risk that one party will not fulfill its obligations under a financial contract which would result in other party's financial loss (OECD, 2005). As such, credit risk results from any sort of credit extension. More specifically, credit risk can be subcategorized into issuer or counterparty risk.
 - a. Counterparty risk – is the risk that a borrower will not be able to fulfill his obligations under a contract. Ex-ante transaction, counterparty risk pertains to the risk of default by counterparty as well as the risk that a financial intermediary would default before the transaction is complete. A financial intermediary in this sense may provide clearing/underwriting services. Ex-post default i.e. when either counterparty or the financial intermediary has defaulted, risk of the transaction moves to estimating losses due to exposures that are not hedged anymore due to default. In such a case, it's important to find a way to hedge the same exposures using a similar contract.
3. Political risk – is the risk that comes from the possibility of a government adversely affecting the returns on investments through its policies (Bekaert & Hodrick, 2007). In times of an economic downturn, a country's government can decide to enforce capital controls to protect its currency from attacks by speculators, among many things. When investments are made in politically unstable countries, this risk should be properly assessed and accounted for.

4. Reputational risk – refers to losses incurred by a business when its reputation gets hurt. Such losses can be sometimes unexpected as it's difficult to assess what kinds of activities may negatively impact the reputation of a firm.
5. Market risk – the volatility in returns arising due to exposure to the overall market (Bekaert & Hodrick, 2007). Market risk can also be referred to as systemic risk.

The above list of risks is still not exhaustive. Firms can have many different kinds of risks based on their specific activities. For instance, profitability of a firm that sells grains takes a direct impact from basis risk, which is the difference between prices of the hedged vehicle and the prices of the asset. For a firm that sells grains, basis risk is the difference between prices of grain futures/forwards and the spot price for grains. Except for instances where price fluctuations in the hedging instrument fully offset the price fluctuations in the asset, firms need to be wary of basis risk and should have sufficient liquidity to deal with losses when the hedge expires.

In the past decade, advances in financial innovation have made it relatively easier to hedge risks. However, financial innovations like derivatives come with their own risks. Although derivative instruments allow risk mitigation, inadequate understanding of how these instruments work can multiply losses for a firm. Many firms today hold derivative securities as part of the overall portfolio chiefly to boost portfolio returns as well as insure the portfolio against certain events. But derivatives have their own investment characteristics that require sophisticated methods and models for proper analysis. With all the complexities of today's financial markets, it has become a necessity to use risk modeling to predict extreme losses. In the absence of an adequate risk control system, it can be difficult for firms to get rid of fluctuations in returns.

The purpose of this paper is to analyze three risk models that have become synonymous with risk management. The paper attempts at analyzing Value at Risk (VaR), Expected Shortfall (ES), and Expected Value Theory (EVT) using a hypothetical equity portfolio of five stocks. The paper highlights the main differences between these risk models by applying each of the risk models to the portfolio to estimate maximum losses.

B. Why VaR?

VaR has become a standard tool in risk management. It has become especially important following the 1995 Basel Accord amendment which encouraged banks and deposit-taking-institutes to forecast daily VaR using internal models (Chang, Jimenez-Martin, McAleer, & Perez-Amaral, 2011). Before VaR, risk was measured in terms of standard deviation. The popularity of Markowitz portfolio theory prompted investors to think of investments in terms of excess returns per unit of risk. At that time,

standard deviation seemed like a suitable measure of risk. However, there are a few problems with using standard deviation as a measure of investment risks (Brouwer, 2001).

- a. Behavioral finance supporters argue that an investor does not perceive of risk in terms of volatility. Investors are only concerned about downside volatility whereas standard deviation is a symmetric measure i.e. it treats losses and gains the same way.
- b. The assumption that variance, square of standard deviation, is finite is contrary to what financial mathematics has proven. Experiments suggest that second moment is non-existent in financial markets.
- c. Central limit theorem applies only to the center of the distribution and thus does not explain the tails of the distribution, the area which exposes an investor to risks.
- d. Many financial instruments cannot be explained by the Gaussian distribution. For instance, options, warrants etc have non-normal distributions.
- e. Although not as sensitive to mean or range, variance is an unstable measure that is also influenced by the presence of outliers.

These issues with the use of variance for risk prediction led to a more stable and reliable risk measure – VaR. VaR is an estimate of the amount of loss that can be incurred on a given portfolio over a specific time horizon T with a confidence level $(1-\alpha)$. VaR assumes that repeated sampling results in a small probability α of losses exceeding VaR for a given horizon T (Brouwer, 2001). Thus, VaR provides an estimate for a possible dollar-loss on a portfolio and makes it easier to compare *riskiness* across investment portfolios.

C. Hypothetical Equity Portfolio (2009 to 2012)

The hypothetical portfolio is composed of daily equity returns for five stocks namely, BP, HP, Guess, Google, and Ford. The period chosen for this hypothetical portfolio was 2009 – 2012, right after the financial crisis. The selection of period over which this hypothetical portfolio existed is a factor that could affect the results of loss estimates using the three risk models. Period representativeness is key, as the period should be reflective of actual risks associated with each stock. Because of this reason, three years of historical data was used in modeling risks for each stock and subsequently the portfolio. If we had selected 2000 – 2012 as the period, one could argue that there was a regime shift during this period. Regime shift means a persistent change in the underlying structure of a system due to external factors. The period would not have been representative of each stock, as the factors that influenced stock performance prior to the financial crisis may be different post crisis. Regression techniques can be applied to test for regime shifts, but we will not concern ourselves with that in this paper.

1. Estimating VaR Using the Historical Method

In this paper, the two methods used to calculate VaR are

- a. Historical method
- b. Simulation approach

Historical method uses historical data to estimate VaR over a period t with a specific confidence interval. For the analysis of the portfolio and each of its individual securities, VaR estimates were calculated with 95% and 97.5% confidence intervals. Horizon t used in this case is a one-day period. For a normal distribution, confidence intervals corresponding to these two values capture about 95% and 97.5% of the entire return distribution. Because we are only concerned with the frequency and level of extreme losses in the tails of the distribution, 95th and 97.5th percentiles of the distribution provide an estimate for VaR such that there is 5% and 2.5% probabilities of losses exceeding VaR, respectively.

Table 1 below provides summary statistics for individual stocks and the portfolio. Over the 2009 – 2012 period, Ford's stock had the highest mean return while HP's stock had the lowest. The standard deviations for all five stocks and the portfolio range between 26.15% (for the portfolio) to 44.39% (for GUESS's stock). Kurtosis for the six entities is positive, which means that return distributions are leptokurtic i.e. they have fatter tails than what is suggested by the normal distribution. Returns are clustered more around the mean, which gives distributions peakedness and fatter tails. The kurtosis for GUESS is 3.30, which is quite close to the kurtosis for a normal distribution, which is 3.0. Hence the return distribution for GUESS can be assumed to be close to normal. Kurtosis of 2.54 for the Ford's stock also indicates a return distribution close to normal. The overall portfolio also has positive kurtosis, but it is less than 3.0. Thus the overall portfolio's return distribution is platykurtic, which means that it has slightly thinner tails than a normal distribution.

	VaR - 95% confidence Interval	VaR - 97.5% confidence Interval	Mean	Standard Deviation	Kurtosis
HP	-3.19%	-4.14%	-32.41%	32.51%	20.86
BP	-2.99%	-4.07%	-4.77%	34.57%	11.73
GUESS	-4.61%	-5.66%	7.37%	44.39%	3.30
FORD	-3.84%	-4.65%	23.22%	38.65%	2.54
GOOG	-2.42%	-3.33%	15.88%	26.96%	8.67
Portfolio	-2.78%	-3.56%	1.08%	26.15%	2.03

Table 1: Portfolio and Stocks' statistics with VaR estimates

For distributions that are leptokurtic, VaR underestimates the true probability of losses exceeding a certain threshold. For instance, HP's stock has the largest kurtosis, yet its VaR estimates are comparable

to that of Guess and Ford, which are almost normally distributed. Figure 1 shows the histogram for HP's stock.

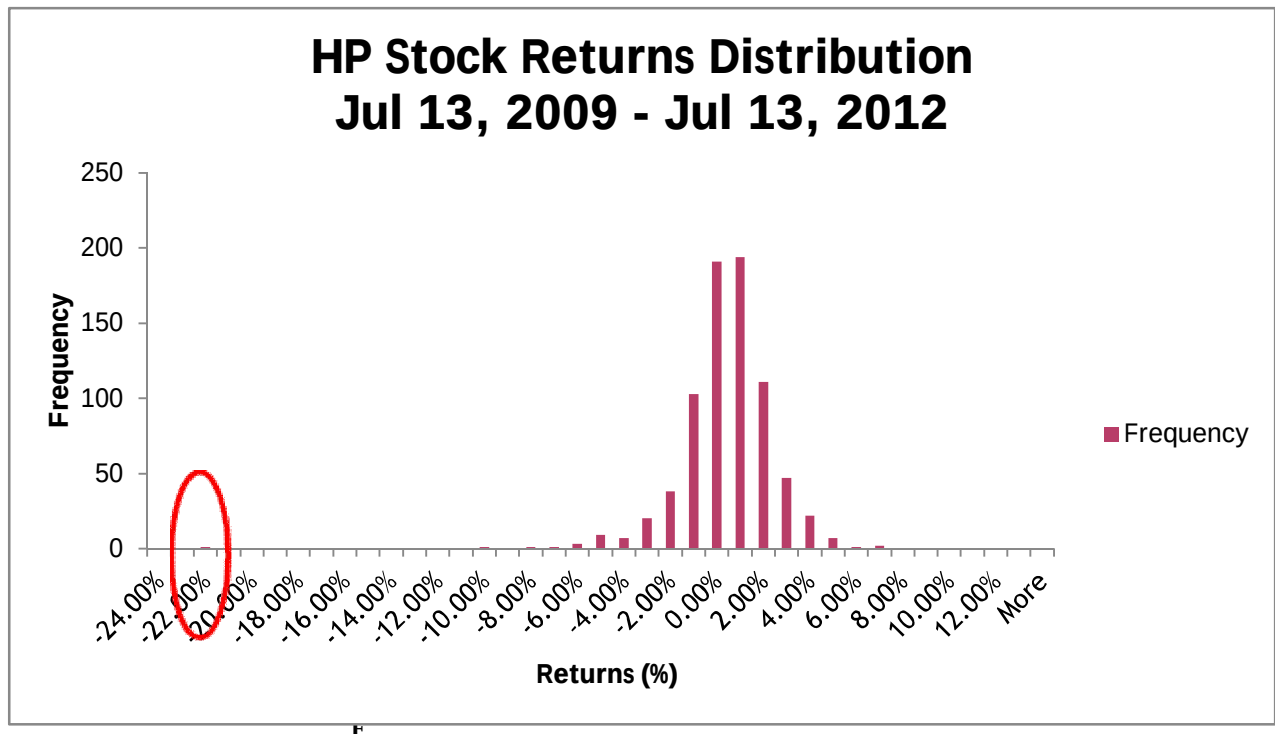


Figure 1: HP Stock Return Distribution with fat tails

As can be seen, returns are more frequent close to the mean and the distribution has extreme returns in the tails. VaR for HP's stock with a 95% confidence is -3.19%. Even with a 97.5% confidence interval, VaR predicts a -4.14% loss. However, the stock had daily returns as low as -22%, as the histogram also indicates. Thus caution needs to be taken when interpreting VaR results for different assets. If the asset returns do not follow a normal distribution, which is one of the key assumptions used in VaR calculations, then the estimates for maximum losses with a specific confidence interval will tend to underestimate the risks and may lead to faulty investment decisions.

Problems with Historical Method for VaR Calculations

One of the main advantages of Historical Method for VaR calculations is that it is easy to implement. A large data set is the only requirement for VaR calculations under this approach. At the same time, reliance on empirical data is also a pitfall of this approach. The data used in the case of this hypothetical portfolio consists of a period that had unusual volatility in asset returns, which may not be representative of the individual stocks. The U.S. economy was coming out of recession and moving toward a slow recovery. It was an exceptional period in the history of financial markets as a result of which VaR estimates for each individual security as well as the portfolio may under or overestimate

actual risks. It's always a challenge to select a sample set that is representative of the underlying population, and the Historical Method exacerbates that

problem. The tradeoff involved is between the implementation convenience of the model and the accuracy of the sample statistic, which in our case is the VaR estimate with 95% and 97.5% confidence intervals.

The Basel Committee on Banking Supervision pointed out the dilemma of Historical Method as, "most risk management models, including stress tests, use historical statistical relationships to assess risk. They assume that risk is driven by a known and constant statistical process, i.e. they assume that historical relationships constitute a good basis for forecasting the development of future risks. The crisis has revealed serious flaws with relying solely on such an approach" (Bank of International Settlements [BIS], 2009). Although majority of the models rely on historical values to some extent for calculations of sample statistics, complete and sole reliance on past data for forecasting of risk measures like VaR can lead to faulty risk estimates.

An alternative to the Historical Method for calculating VaR is the Historical Simulation approach. The advantage to using Historical Simulation approach is that it uses a monte carlo approach to simulate sample returns for an underlying population. By doing this, noise that often exists in the original data set is removed so that VaR calculated using the randomly generated sample reflects true risks associated with an asset.

2. Historical Simulation

In order to apply this method, sample returns were simulated using sample standard deviation and mean of the stocks over the 2009-2012 period. In order to compare VaR values across methods, a total of 759 observations were simulated using the random number generator in order to match the size of the data set to that used in the Historical Method approach.

Continuing with the BP example, the difference between the VaR values using the Historical Method versus the Historical Simulation method is apparent in Table 2. Although both the methods make use of the historical data, the estimated Value at risk for the BP stock is slightly larger in magnitude with the Historical Simulation method than with the Historical Method. The small difference between the VaR values at 95% confidence and 97.5% confidence intervals using the Historical Simulation Method suggests smaller variability in the sample than what is being suggested by the Historical Method.

Historical Simulation				
Initial Investment in Portfolio	95th Percentile		97.5th Percentile	
	%	in \$	%	in \$
\$1,000.00	-2.83%	-\$28.30	-3.37%	-\$33.70
Historical Method				
Initial Investment in Portfolio	95th Percentile		97.5th Percentile	
	%	in \$	%	in \$
\$1,000.00	-2.78%	-\$27.80	-3.56%	-\$35.60

Table 2: Comparison of VaR estimates for the Portfolio under the Historical and the Historical Simulation approaches

The simulation approach predicts a larger daily loss for the portfolio at both 95% and 97.5% confidence intervals. However, we know from our discussion above that the portfolio has a kurtosis smaller than 3 i.e. 2.03. Because the return distribution for the portfolio is platykurtic (thin tails), the standard deviation for the empirical data is larger than it would be if the portfolio return distribution had a kurtosis of 3.0 or greater, as data points are more spread out in a platykurtic distribution. In order to simulate returns for the portfolio, standard deviation and mean from the empirical data series were used as inputs. Thus, VaR estimates under the simulation approach overestimate portfolio risks due to their reliance on a standard deviation that comes from a platykurtic distribution. The simulation approach assumes, without regards to the kurtosis for the distribution, that there are more chances of having large losses in the tails of the distribution due to a standard deviation that indicates a more spread out data set. Sample representativeness thus again becomes an issue in the simulation approach as it uses standard deviation and mean from empirical data to simulate returns for the same period.

Criticisms against VaR

Consider a trader at a bank who found a deal which would yield 35% in the best scenario, but based on his monthly VaR calculation for this position with a 95% confidence interval, the losses could be as much as 22% if the markets move in the other direction. The trader wants to carry out this trade with 40% of his \$100 million portfolio. Thus, the anticipated loss on a \$40 million position is \$8.8 million using a 95% confidence interval, and there is only a 5% chance of losses exceeding this level. Due to a historically small standard error, which is the standard deviation of a sampling distribution of a statistic, both the firm and the trader strongly believe that VaR estimate is accurate. In spite of the bank having enough capital at its disposal to sustain losses of a magnitude of \$8.8 million over a period of a month, the markets unexpectedly moved in the other direction. Instead of \$8.8 million losses, total loss incurred on the position over a month is a staggering 30% of the overall \$40 million position i.e. \$12 million. The

unexpected \$3.2 million in losses above the expected \$8.8 million severely affected the bank's liquidity position. Investments with significantly high expected returns also have high risks.

The above example exhibits one of the greatest shortcomings of VaR, which is that it fails to explain magnitude of losses in extreme events. The word *extreme* takes on a whole new meaning in the context of risk management. VaR is a great tool to get an intuitive sense for capital that can be at risk in a certain transaction, but it assumes asset returns are normally distributed. If this assumption is violated and individual asset returns are in fact non-normally distributed then VaR estimates will be misleading, as they will underestimate the true capital at risk (Sollis, 2009). Thus, there is a need to explore risk models that are more robust than VaR i.e. a violation of one of their assumptions does not alter their results.

One of the main criticisms against VaR is that it's not a coherent risk measure. A coherent risk measure is one which exhibits the following five properties.

For a probability space $(\Omega, \mathcal{F}, P)$ and a time horizon denoted by Δ , there is a set of random variables on $(\Omega, \mathcal{F})$, represented by $L^0(\Omega, \mathcal{F}, P)$, that is finite. Then a set $M \subset L^0(\Omega, \mathcal{F}, P)$ is a set of portfolio losses over a time horizon Δ . M can be seen as a convex cone satisfying the following properties (Embrechts, Frey & McNeil, 2005)

- a. $L_1 + L_2 \in M$
- b. $\lambda L_1 \in M$ for $\lambda > 0$

Any risk measure $q: M \rightarrow \mathbb{R}$, where M is the domain of q and $\mathbb{R}$ is the co-domain, needs to satisfy the following four properties in order to be coherent.

- a. Translation Invariance for all $L \in M$ and every $l \in \mathbb{R}$,

$$q(L+l) = q(L) + l$$

This property states that adding or subtracting a deterministic quantity l to a position that led to loss L should lower the risk of the portfolio by reducing capital requirements exactly by the same amount. For instance, if a portfolio had a loss $L > 0$, adding an amount of capital $q(L)$ will lead to the adjusted loss $L' = L - q(L)$, so that there is no need for an extra capital injection.

- b. Sub-additivity for all $L_1, L_2 \in M$

$$q(L_1+L_2) \leq q(L_1) + q(L_2)$$

The above statement reflects that if a portfolio is diversified, the risk of the overall portfolio should be less than the individual component risks. It is an important requirement to have for any good risk measure should have. Subadditivity also allows for decentralization of risk. For instance, if positions of two trading desks lead to losses L_1 and L_2 , a risk manager can use a

subadditive measure to ascertain that $q(L_i)$, the overall risk of the portfolio is less than some number M .

- c. Positive homogeneity for all $L \in M$ and $\lambda > 0$,

$$q(\lambda L) = \lambda q(L)$$

It means that a change in the size of a portfolio should only result in a proportional change in capital requirements, as there is no change in the composition of the portfolio.

- d. Monotonicity for $L_1, L_2 \in M$, such that $L_1 \leq L_2$, we have

$$q(L_1) \leq q(L_2)$$

The axiom of monotonicity shows that for positions with smaller losses, the capital requirements should also be lower.

Generally VaR does not satisfy the subadditivity requirement and thus is not a coherent risk measure. VaR fails to be subadditive when the assets in a portfolio have highly skewed loss distributions, which is usually the case for portfolios composed of options, warrants, high yield bonds etc. Even when assets have smooth and symmetric distributions but their copulas (explain dependence structures between random variables) are asymmetric, VaR becomes non-subadditive. VaR is also non-subadditive when an underlying random variable has a leptokurtic or a heavy-tailed distribution. VaR is only subadditive in the ideal scenario where all given portfolios can be expressed in terms of linear combinations of the same underlying elliptically distributed risk factors (Embrechts et al., 2005).

As can be seen, VaR is not a very robust risk measure as one can infer reliably about the losses for a given portfolio using VaR only under a very special scenario. In real world, portfolios are not composed that ideally. It is more than likely to have portfolios containing derivative instruments to some extent, in which case VaR estimates can lead to not so optimal investment decisions.

Another problem with Var is that it reveals nothing about the magnitude of losses in case of extreme events and the probabilities associated with them. In order to infer more accurately magnitude of losses under extreme events, Expected Shortfall can be used either in conjunction with VaR or some other risk model, such as Extreme Value Theory (EVT), to estimate the degree of losses and probabilities associated with events causing them.

3. Expected Shortfall (ES)

Besides being a coherent risk measure, Expected Shortfall method also has the advantage that it provides easy implementation and relatively stronger interpretative power than VaR. Under an ES approach, average losses in the tails of a distribution are taken at a specified confidence interval. The confidence intervals used in this case are the same as the ones used to calculate VaR i.e. 95% and

97.5%. Table 3 gives the ES estimates for all five stocks as well as the portfolio. It can be seen that ES estimate for the overall portfolio is smaller compared to individual stocks. This is one of the main properties of a coherent risk measure that it encourages diversification to reduce overall risk for the portfolio, which is in line with Markowitz's modern portfolio theory.

	95% confidence Interval	97.5% confidence Interval
GOOG	-3.99%	-5.16%
BP	-5.22%	-6.95%
HP	-5.27%	-6.86%
GUESS	-6.44%	-7.89%
FORD	-5.26%	-6.22%
PORTFOLIO	-3.97%	-4.79%

Table 3: Average losses at 95% and 97.5% confidence intervals for individual stocks and the portfolio

To carry out an ES analysis, sample returns first need to be arranged in ascending order and then average of the lowest 5% and 2.5% of the returns yields ES estimates with a 95% and 97.5% confidence intervals respectively. For a sample of 759 observations, it corresponded to the lowest 38 observations for the 95% confidence interval and the lowest 19 observations for the 97.5% confidence interval. Hence for the portfolio under consideration, there is a 5% probability that losses could exceed 3.97% and a 2.5% probability that losses could more than 4.97% of the capital invested in the portfolio. Compared to the Historical and Simulation approaches to calculating VaR, ES resulted in a higher estimate for daily VaR for the portfolio, although the portfolio's return distribution is platykurtic. This difference in estimates for daily losses is due to the fact that ES is the *average* of the daily losses that may occur in the tails whereas VaR is simply the p^{th} percentile for a return distribution. VaR does not factor in the magnitudes of losses above the p^{th} percentile while ES does.

Figure 2 shows the trend in ES estimates at 95% and 97.5% confidence intervals. It can be noted how diversification significantly lowered average losses incurred during extreme events for a multi-asset portfolio.

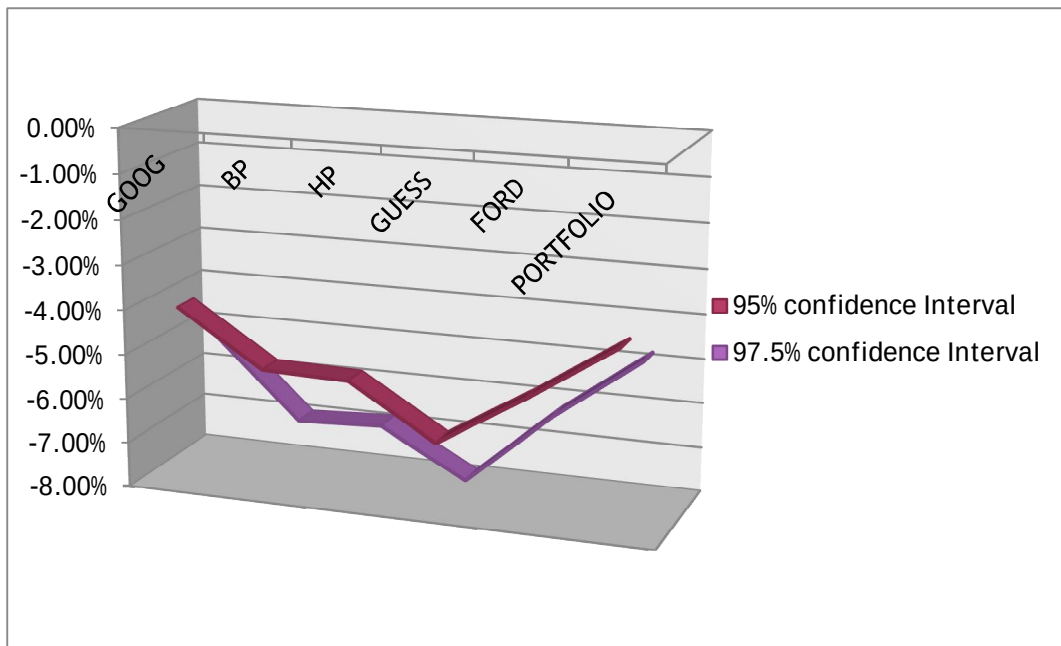


Figure 2: ES values for individual stocks and the hypothetical portfolio at 95% and 97.5% confidence interval

Alternative to ES – Expected Value Theory (EVT)

Risk models discussed so far serve the purpose of quantifying maximum losses a portfolio could suffer at a $(1-\alpha)$ confidence interval, where α is the significance level. The disadvantages of using VaR have been discussed in an earlier section, and as an alternative to VaR, ES method was introduced. It was showed how ES is a coherent risk measure and therefore is a more desirable risk measure than VaR. But ES still doesn't explain the probabilities associated with extreme events on its own. In order to adequately quantify losses in case of an extreme event, risk modeling of the tails of a distribution is necessary. For this purpose, Expected Value Theory (EVT) can be combined with VaR or ES methodologies to analyze the tail behavior and estimate extreme losses that occur with a very low frequency. Before carrying out EVT analysis, it is worthwhile to observe quantile-quantile plot for the data set in question.

Quantile – Quantile (Q-Q) Plot is a visual tool that shows the relationship between empirical data and reference distribution by plotting quantiles of empirical data against quantiles of a reference distribution (Embrechts, Frey, McNeil, 2005, pp. 68-69). Figure 3 shows the Q-Q plot for the five asset portfolio by plotting quantiles of the portfolio return distribution against quantiles of a normal distribution. Lack of perfect linearity in the plot indicates departures from the normal distribution.

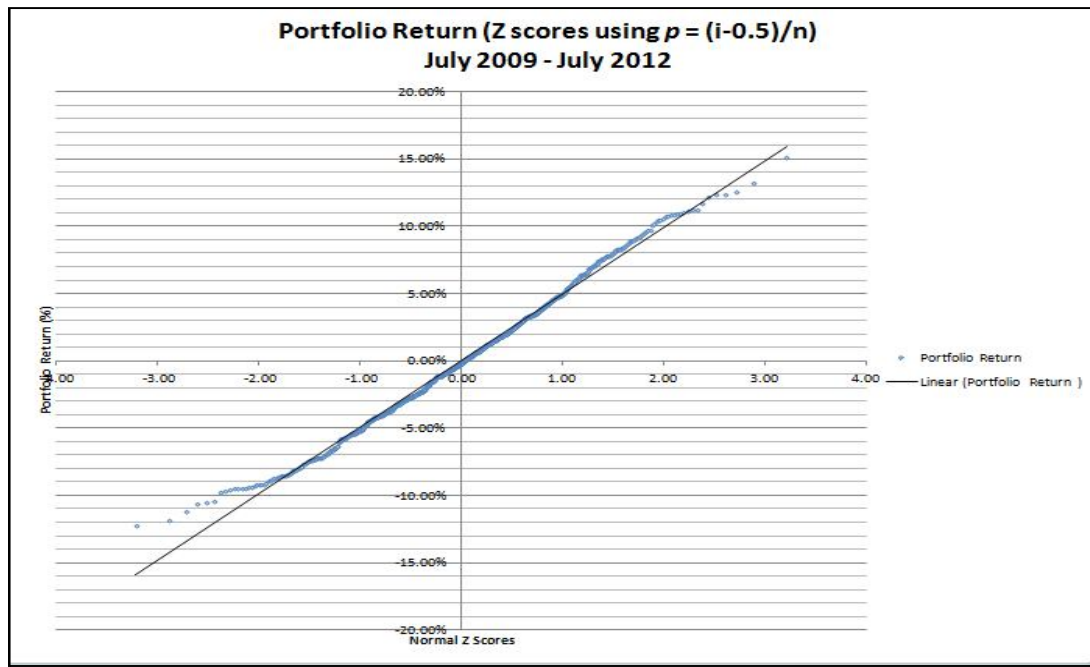


Figure 3: Q-Q plot for the portfolio

When empirical data series follow a normal distribution, q-q plot indicates almost a perfect linear relationship between quantiles from the data set and those derived from the standard normal distribution. In the figure above, quantiles on the bottom left and top right do not align linearly with the theoretical quantiles for a normal distribution. In fact they drift away from the linear line, which indicates that portfolio returns are not perfectly normally distributed, and the distribution may have fat tails. As the portfolio return distribution has a kurtosis of 2.03, we know that there is a departure from normality as the distribution has thinner tails than a normal distribution. In this case, VaR and ES estimates may have overstated true risks for the portfolio. Hence, QQ plots serve as a validation tool in EVT analysis and help in making the right distributional assumptions based off of which EVT analysis can be performed.

EVT analysis

There are two main methods that are used to apply EVT to a data set – Block Maxima method and the Peaks-Over-Threshold method. First we discuss the Block Maxima approach.

- a. GEV Approach to Block Maxima is a method used to estimate a distribution F that lies in the domain of attraction of an extreme value distribution H_ε for some ε , formally denoted as $F \in \text{MDA}(H)$. If some normalized block maxima M_n , where $M_n = \max(X_1, X_2, \dots, X_n)$, converges in distribution as $n \rightarrow \infty$, then H_ε must belong to the GEV family of distributions. The parametric form for the GEV family of distributions is

$$H_\varepsilon(x) = \exp(-(1 + \varepsilon x)^{-1/\varepsilon}), \quad \varepsilon \neq 0$$

$$\exp(-e^{-x}), \quad \varepsilon = 0$$

Depending on the value for ε , H_ε can take any one of three forms under the GEV parametric form. If $\varepsilon = 0$, the distribution is Gumbel. If $\varepsilon > 0$, the distribution is a Frechet distribution. If $\varepsilon < 0$, the distribution is a Weibull distribution (Embrechts et al., 2005).

The key point of the GEV method is that if there are n -block maximum M_n , then the true distribution of the underlying can be approximated using a three parameters GEV distribution, where the parameters involved are μ , ε , and σ . Using the block maxima method, one needs a large number of observations for maxima which are drawn from a large number of samples with iid rvs (random variables). The block maxima method analyzes the maximum or minimum values for a given distribution of n -block maxima (Liow, 2008). The data is first divided into n non-overlapping blocks and the maxima observations are fitted to a GEV distribution.

This approach was, however, not used to analyze the tail distribution for the portfolio under consideration. Although more straightforward than the alternative peaks-over-threshold (POT) method, a downside of this method is the need for an extremely large data set that can then be divided into large blocks in order to find the maxima corresponding to each block. In reality, however, maxima don't occur with high frequencies. Thus, in the absence of large blocks of maxima, the assumption that n -block maxima converges to a GEV parametric form will be erroneous and the conclusions regarding the maximum value for an underlying distribution will be less reliable.

- b. GPD Approach to Peaks-Over-Threshold is a modern alternative to the block maxima approach. It can also be used for large observations, and in fact, it uses the limited number of extreme observations more efficiently than block maxima. In POT analysis, a threshold u is chosen such that observations that exceed the threshold are analyzed as a tail distribution. Two approaches that can be employed for POT analysis are (McNeil, 1999):

1. Semi-parametric approach – Hill estimator is one of the most studied methods for the modeling of heavy-tailed distributions. The method makes an assumption about a loss distribution as being in the domain of attraction of the Frechet distribution from the GEV family. Under this assumption, tail of the loss distribution has a form of

$$F(x) = L(x) x^{-\alpha}$$

The Hill method is best for distributions that decay at a slowing rate as the underlying assumption assumes the loss distribution to be a Frechet distribution which decays at a very slow rate and has an infinite upper tail (Embrechts et al., 2005). Hill method is more focused on the tail index, α , which is the reciprocal of the shape parameter ε . The selection of the

tail index needs to be such that optimizes the bandwidth of the loss distribution i.e. the range of values between the upper and lower bounds for a tail distribution (Elroy, 2007). Because the hill method is semi-parametric, there is a trade-off between the ease of implementation and the efficiency with which the data is employed. It's, however, more straightforward than POT or block maxima methods, but if the underlying loss distribution is not in the maximum domain of attraction of a Frechet distribution then the inferences based on this method will be incorrect.

2. Peaks Over Threshold (POT) is a parametric model that is used in extreme value theory to model extreme observations as a Generalized Pareto Distribution (GPD) (Embrechts et al., 2005). The distribution function for a GPD is given by

$$G_{\varepsilon,\beta}(x) = 1 - (1 + \varepsilon x / \beta)^{-1/\varepsilon}, \quad \varepsilon \neq 0$$

$$1 - \exp(-x/\beta), \quad \varepsilon = 0$$

where ε and β are the shape and scale parameters respectively. For $\varepsilon > 0$, the distribution function $G_{\varepsilon,\beta}(x)$ is an ordinary Pareto distribution and has a heavy tail. When $\varepsilon = 0$, the distribution function is that of an exponential distribution. When $\varepsilon < 0$, the distribution function is that of a Pareto type II distribution with a short-tail. In the GPD family of distributions, the shape parameter guides the selection of a distribution to be used for modeling tail of an empirical data set. For the purpose of modeling the losses of the hypothetical portfolio, POT method was used. First step in POT method requires the selection of a threshold u such that there are sufficient numbers of observations that exceed the threshold. Threshold should be selected such that small changes in its values don't result in significant changes in the values of the estimated quantiles. It is therefore important to select a value for the threshold in a region where the tail behavior of the data being analyzed is stable and does not exhibit high volatility (Jaruskova & Hanek, 2006).

Observations above a threshold u are assumed to have a distribution function $F(y)$ (McNeil, 1999) such that,

$$F_u(y) = P \{X - u \leq y \mid X > u\}$$

$F_u(y)$ represents the conditional probability of a loss exceeding threshold u by an amount equal to y , given the fact that variable X exceeds the threshold. In practice, it is mostly assumed that the underlying loss distribution probability function $F_u(y)$ has an infinite right end-tail, which helps make a simplifying assumption that there is a possibility of having extremely large losses, even if the probability associated with those outcomes is negligibly small. An important theorem regarding the loss distribution $F_u(y)$ is that for a large class of

underlying distributions F , as the threshold u is increased, F_u converges to a generalized Pareto distribution (McNeil, 1999). This large class of underlying distributions includes all of the statistical and actuarial distributions such as Normal, beta, chi-square (χ^2), t , F , etc. Another important point is that the underlying loss distribution F on an interval $(0, 1]$ is an example of a distribution with a finite right end point, in which case there is zero probability associated with losses above 1. Such distributions can be helpful in analyzing credit related losses.

Mean Excess Plot is a visual representation of the excesses above a threshold u . For the portfolio in point, absolute losses (portfolio returns < 0) were arranged in descending order, and excesses above the thresholds were calculated for different threshold values starting from 0.5% to 8% at an interval of 0.5%. For each threshold, average of the excesses was calculated and plotted against threshold values. Figure 4 below shows a graph of mean excesses as a function of threshold u .

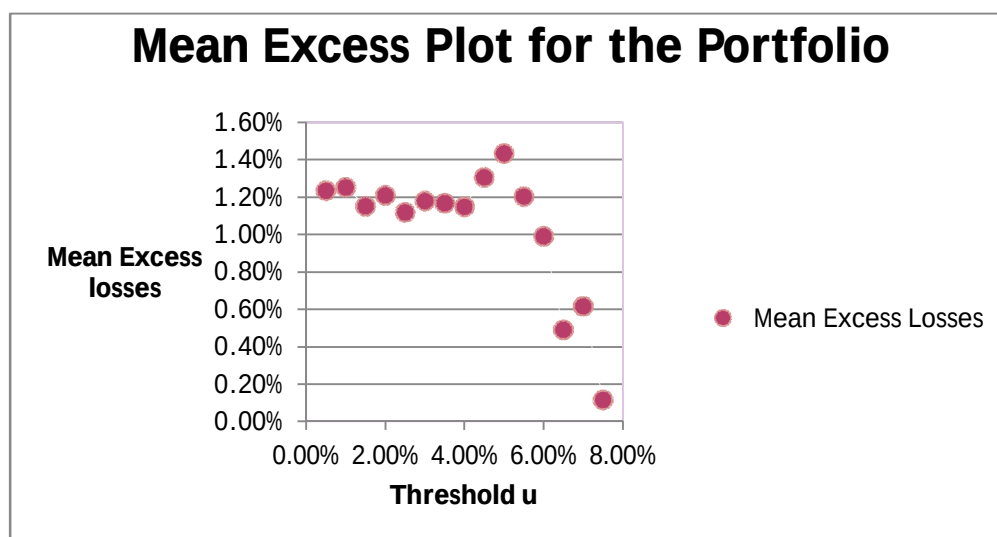


Figure 4: Mean Excesses for the multi-asset portfolio

As the plot shows, mean excesses exhibit a random pattern. However, close to threshold value 2.5%, mean excesses vary linearly with threshold u . Further the standard errors, which are the standard deviations of sampling statistics, for different threshold values show that threshold equal to 2.5% also has a small standard error. Even though thresholds 1.5% and 2.00% have the lowest and second lowest standard errors, those values for threshold are not selected because mean excesses in that region of the mean excess plot do not vary linearly with the threshold values. Therefore, 2.5% was selected as the threshold to be used in POT analysis of the portfolio. Figure 5 below provides standard errors and confidence intervals for different threshold values.

Mean Excess Losses	Threshold, u	Standard Error Of Mean Excess Loss	Upper Limit	Lower Limit
1.24%	0.50%	0.0771%	1.31%	1.16%
1.25%	1.00%	0.0943%	1.35%	1.16%
1.15%	1.50%	0.1106%	1.26%	1.04%
1.21%	2.00%	0.1394%	1.35%	1.07%
1.12%	2.50%	0.1595%	1.28%	0.96%
1.18%	3.00%	0.2036%	1.38%	0.98%
1.17%	3.50%	0.2507%	1.42%	0.92%
1.15%	4.00%	0.3012%	1.45%	0.85%
1.31%	4.50%	0.3853%	1.69%	0.92%
1.44%	5.00%	0.3902%	1.83%	1.05%
1.20%	5.50%	0.3641%	1.57%	0.84%
0.99%	6.00%	0.3168%	1.31%	0.68%
0.49%	6.50%	0.3168%	0.81%	0.18%
0.62%	7.00%	na		
0.12%	7.50%	na		

Figure 5: Standard errors and confidence intervals for varying threshold values

The distribution of excesses above a threshold u is given as (Bensalah, 2000):

$$F(u+y) = P(X-u \leq y \mid X > u) = P(Y \geq y \mid X > u), y \geq 0$$

Given the above distribution, the distribution function for extreme observations, X_i follows:

$$F(u+y) = P(X \geq u+y) = P((X \geq u+y \mid X > u) \cdot P(X > u))$$

$$F(u+y) = P((X-u \geq y \mid X > u) \cdot P(X > u))$$

$$F(u+y) = F_u(y) \cdot F(u)$$

A Pickands, Balkema-in Haan theorem suggest that for a high threshold,

$$(F_u(y)) \approx G_{\varepsilon, \beta(u)}(y)$$

The distribution function $F(u)$ can be estimated as

$$F(u) = (1/n) \sum I_{\{X_i > u\}} = (N_u/n)$$

Substituting, we get

$$F(u+y) = (N_u/n)[1 + \varepsilon(y/\beta)]^{-1/\varepsilon}$$

where y = excess above threshold

ε = shape parameter

β = scale parameter

Using 2.5% as the threshold, maximum likelihood estimation (MLE) was used to select the values for the shape and scale parameters. MLE involved maximizing sum of the log of the probability distribution function of the mean excesses for different values of the shape parameter. A value of 0.25

was found for the shape parameter value, ε , with a scale parameter value of 1.05. Next the quantiles for the distribution function of extremes were calculated using

$$F(u+y) = (N_u/n)[1 + \varepsilon(y/\beta)]^{-1/\varepsilon}$$

After calculating the quantiles, VaR with 95% and 97.5% confidence intervals is calculated, which resulted in

	VaR with 95% confidence	VaR with 97.5% confidence
Portfolio	6.2915%	6.4212%

Figure 6: VaR for the portfolio under the EVT approach

In comparison to the VaR values obtained using the Historical and Expected shortfall approaches, EVT approach yielded estimates for value at risks that are significantly higher than VaR for either of the two methods. It should be noted that the values for VaR in figure 6 represent absolute losses due to which they are positive.

EVT focuses on modeling tail of a distribution, and thus this method attempts at capturing losses in highly unlikely events. In order to be able to inject sufficient capital to sustain losses during times of crisis, one needs a model that can estimate those extreme losses with a high probability. Combining the VaR approach with EVT provides solution to this problem.

Concluding remarks

In this paper, using the hypothetical equity portfolio as the basis for analysis, extreme losses were estimated using four different methods – Historical approach to the calculation of VaR, Simulation approach, Expected Shortfall, and EVT applied under the POT approach. ES, Historical method, and Simulation approach to calculating VaR underestimated extreme risks that the portfolio was exposed to on a daily level over the 2009 – 12 period, as these methods make an assumption of normality in asset returns. In practice, assets are part of a larger portfolio that is a blend of various securities that may or may not have normally distributed returns. Thus it becomes difficult to obtain an accurate estimate for capital at risk for an overall portfolio under the ES and VaR approaches when the normality assumption does not hold. EVT, on the other hand, is a specialized framework for analyzing losses in extreme events that occur with a small probability. It's a parametric model that has the flexibility to adapt to different data sets based on their particular parameter estimates. Hence, in the special case of extreme loss analysis, EVT is a more preferred method than either ES or VaR approaches and should be part of every risk manager's toolkit.

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