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Transcendental Numbers and Infinities of Different Sizes

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Abstract

There are many different number systems used in mathematics, including natural, whole, algebraic, and real numbers. Transcendental numbers (that is, non-algebraic real numbers) comprise a relatively new number system. Examples of transcendental numbers include e and π. Joseph Liouville first proved the existence of transcendental numbers in 1844. Although only a few transcendental numbers are well known, the set of these numbers is extremely large. In fact, there exist more transcendental than algebraic numbers. This paper proves that there are infinitely many transcendental numbers by showing that the set of real numbers is much larger than the set of algebraic numbers. Since both of these sets are infinite, it means that one infinity can be larger than another infinity.

Background

Set Theory

In order to show that infinities can be of different sizes and to prove the existence of transcendental numbers, we first need to introduce some basic concepts from the set theory. A set is a collection of elements, which is viewed as a single object. Every element of a set is a set itself, and further each of its elements is a set, and so forth. A set A is a subset of a set B if each element of A is an element of B, which is written $A \subset B$. The smallest set is the empty set, which has no elements and denoted as $\emptyset$. Thus the empty set is a subset of every set. The power set of A is the set whose elements are all the subsets of A, and it is denoted as $\mathcal{P}(A)$. For example, if $A = \{2, 5\}$, then $\mathcal{P}(A) = \{\emptyset, \{2\}, \{5\}, \{2, 5\}\}$ The cardinality of a set is the number of elements in a finite set, and denoted as $|A|$. A finite set A with n elements has 2^n subsets, that is $|\mathcal{P}(A)| = 2^n$ Two different finite sets are of the same size if they have the same number of elements. For example, $A = \{1, 4, 7\}$ and $B = \{2, 4, 10\}$ have the same size, in other words, $|A| = |B| = 3$.

There are certain operations that can be performed on sets. The intersection of A and B is the set of all the elements that are in both A and B and is denoted as $A \cap B$. The union of A and B is the set of all elements that are in A or in B, and is denoted as $A \cup B$. Sets A and B are called disjoint if they have no common elements, in other words, $A \cap B = \emptyset$. The complement of the set A is the set of all the elements that are not in A, and is written as A^c.

Number Systems

All systems of numbers are sets. Set of natural numbers $\mathbb{N}$ consists of all positive whole numbers. $\mathbb{Z}$ is the set of integers and consists of positive whole numbers, negative whole numbers and zero. The set of rational numbers $\mathbb{Q}$ is comprised of real numbers that can be written as a quotient of two integers (where denominator is not equal to zero). $\mathbb{R}$ is the set of real numbers that consists of rational and irrational numbers. The set of algebraic numbers $\mathbb{A}$ consists of real numbers that are roots of some polynomial equations with rational coefficients. And the set of transcendental numbers is comprised of real numbers that are non-algebraic.

Finite and Infinite Sets

Finite Sets

A set A is equinumerous to a set B if there is a one-to-one correspondence between A and B, which is written as $A \approx B$. In this case the set A has the same cardinality as the set B.

Using function terminology, for the set A to be equinumerous to the set B, there must be an injective and surjective function (or mapping) from A to B. Let $f : A \to B$ be a function from the set A to the set B, then the function f is injective, provided that for all $x_1, x_2 \in A$ if $f(x_1) = f(x_2)$, then $x_1 = x_2$. It follows that f is not injective if there exist $x_1, x_2 \in A$ such that $x_1 \neq x_2$ and$f(x_1) = f(x_2)$. A function f is surjective, if for every $y \in B$, there exists an $x \in A$ such that $f(x) = y$. A function f is bijective, if it is both injective and surjective.

For example, let $A = \{1, 4, 7\}$, and $B = \{2, 4, 10\}$, $f : A \to B$ be a function from the set A to the set B, and let f be defined as $f(1) = 2$, $f(4) = 4$, $f(7) = 10$. Clearly, there is a one-to-one correspondence between the set A and the set B. We see that f is a one-to-one mapping onto B. Thus f is bijective.

A non-empty set A is finite if there exists a natural number k such that $A \approx \mathbb{N}_k$, where $\mathbb{N}_k = \{1, 2, 3, ..., k\}$. In this case $|A| = k$. If the set A is a finite non-empty set, then any set equivalent to A is also finite. In other words, finite equinumerous sets have the same cardinality. Furthermore, if the set B is the subset of the finite set A, then the set B is also finite, and $|B| \leq |A|$. Thus no finite set can be equinumerous to any of its proper subsets.

Countably Infinite Sets

A set A is an infinite set if it is not a finite set, that is if there is no bijection between A and $\mathbb{N}_k$. A set A is equinumerous to any of its proper subsets if and only if A is infinite. Also, if a set B is equinumerous to the infinite set A, then B is also infinite. And if the infinite set A is a subset of B ($A \subseteq B$), then B is also infinite. Now consider the set of natural numbers. It consists of the subset of even numbers $\{2, 4, 6, 8, 10, 12, 14, ...\}$ and the subset of odd numbers $\{1, 3, 5, 7, 9, 11, 13, 15, ...\}$.

Natural numbers:	1,	2,	3,	4,	5,	6,	7 ,	8,	9,	10,	11,	12,
	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$
Odd numbers:	1,	3,	5,	7	9,	11,	13,	15,	17,	19,	21,	23, ...

We can define $\mathbb{O}$ to be the set of odd numbers, $\mathbb{N}$ to be the set of natural numbers, and $f : \mathbb{N} \to \mathbb{O}$, defined by $f(n) = 2n - 1$.

First we will show it is injective. Let $n_1, n_2 \in \mathbb{N}$. Assume that $f(n_1) = f(n_2)$.

Then

$$\begin{aligned} 2n_1 - 1 &= 2n_2 - 1 \\ 2n_1 &= 2n_2 \\ n_1 &= n_2 \end{aligned}$$

Thus f is injective.

Now let $y \in \mathbb{O}$. Then

$$f\left(\frac{y+1}{2}\right) = 2\left(\frac{y+1}{2}\right) - 1 = y + 1 - 1 = y$$

Thus we see that f is surjective. Hence f is bijective and it follows that $\mathbb{O} \approx \mathbb{N}$. We see that the set of natural numbers is equinumerous to its proper subset of odd numbers, and so it is an infinite set. The cardinality of $\mathbb{N}$ is denoted by $\aleph_0$. If there is a one-to-one correspondence between an infinite set A and $\mathbb{N}$, then A is a countably infinite, that is, $A \approx \mathbb{N}$ and $|A| = \aleph_0$.

The set of integers $\mathbb{Z}$ is equinumerous to the set of natural numbers $\mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{Z}$ defined by:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{when } n \text{ is even} \\ \frac{1-n}{2}, & \text{when } n \text{ is odd} \end{cases}$$

This function is bijective, thus the set of integers $\mathbb{Z}$ is also countably infinite.
It can be shown that there exist one-to-one mapping from $\mathbb{N}$ onto $\mathbb{Q}$. Hence $\mathbb{Q}$ is also countably infinite.

If both sets A and B are countably infinite, then the union of these sets is a countably infinite set.

Furthermore, let A be a countable set, and let B_n be the set of all n-tuples $(a_1, a_2, ..., a_n)$, where $a_k \in A(k = 1, ..., k)$, and the elements $a_1, a_2, ..., a_n$ need not be distinct. Then B_n is countable (Rudin, 29). In other words, the union of a countable set of countable sets is countable.

An algebraic number is defined to be a number that satisfies a polynomial equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0$$

where the coefficients $a_0, a_1, ..., a_n$ are integers, and $a_n \neq 0$, $n \geq 1$.

We will show that the set of algebraic numbers $\mathbb{A}$ is countably infinite.

First notice that the set of algebraic numbers include all rational and some irrational numbers. For example, $\sqrt{2}$ is an algebraic number, since it is solution to the equation $x^2 - 2 = 0$. And according to the Rational Zeros Theorem, $\sqrt{2}$ is not a rational number. Now let P_n be the set of degree n polynomials with integer coefficients. A polynomial of degree n has $n+1$ coefficients. For example, the polynomial $x^5 - 3x^4 + 11x^2 + 5x + 7 = 0$ is a polynomial of degree 5, and has 6 i.e. $(5+1)$ coefficients: $1, -3, 0, 11, 5, 7$. Such polynomials of degree n can be put into one-to-one correspondence with $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times ... \times \mathbb{Z}$ ($n+1$ times). Thus since the union of countable sets is countable, $f : P_n \to \bigcup_{k=1}^{n+1} \mathbb{Z}_k$ is bijective. Hence P_n is countable. Since a polynomial can have at most n roots, there is a finite number of roots that a polynomial can have. The set of all the possible roots is the union of the roots of all polynomials of degree n. Thus since the union of countable sets is countable, the set of the roots is countable. Hence the set of algebraic numbers $\mathbb{A}$ is infinitely countable.

Uncountably Infinite Sets

If there is no one-to-one correspondence between an infinite set A and $\mathbb{N}$, then A is uncountably infinite. We will show that there is no one-to-one correspondence between the open interval $(0, 1)$ and the set of natural numbers $\mathbb{N}$.

First we see that the open interval $(0, 1)$ contains the infinite subset $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$, that is $\{\frac{1}{n+1} | \, n \in \mathbb{N}\} \subset (0, 1)$. Thus (0, 1) is an infinite set itself.

Now let $f : \mathbb{N} \to (0, 1)$. We will use proof by contradiction to show that there is no bijection between $\mathbb{N}$ and $(0, 1)$. Suppose $f : \mathbb{N} \to (0, 1)$ is surjective. Then we can express the images of all elements in $\mathbb{N}$ in the normalized form:

$$
\begin{aligned}
f(1) &= 0.a_{11}a_{12}a_{13}a_{14}a_{15}... \\
f(2) &= 0.a_{21}a_{22}a_{23}a_{24}a_{25}... \\
f(3) &= 0.a_{31}a_{32}a_{33}a_{34}a_{35}... \\
f(4) &= 0.a_{41}a_{42}a_{43}a_{44}a_{55}... \\
f(5) &= 0.a_{51}a_{52}a_{53}a_{54}a_{55}...
\end{aligned}
$$

$$\vdots$$
$$\vdots$$
$$\vdots$$
$$f(n) \quad = \quad 0.a_{n1}a_{n2}a_{n3}a_{n4}a_{n5}...$$

Now we can construct a real number $b = 0.b_1b_2b_3b_4b_5...$, where $b_1 \neq 9$. We do not allow $b_1 = 9$ in order to exclude the possibility of $b = 1$ (since $1 = 0.99999...$). This number is in the open interval $(0, 1)$, but not on this list. We can define $b = 0.b_1b_2b_3b_4b_5...$, where for every $k \in \mathbb{N}$

$$b_k = \begin{cases} 2, & \text{when } a_{kk} \neq 2 \\ 7, & \text{when } a_{kk} = 2 \end{cases}$$

Since b always differs in the nth decimal place from $f(n)$, $b \neq f(n)$. Thus we constructed a real number b that is in the codomain of f, but not in the image of this function. It follows that f is not surjective, and hence f is not bijective. Thus we have proved that the open interval $(0, 1)$ is uncountably infinite.

Consider $h : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ defined by $h(x) = \tan x$. It is easy to see that h is one-to-one mapping onto $\mathbb{R}$, thus h is bijective.

Now let $f : (0, 1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ defined by $f(x) = (x - \frac{1}{2})\pi$ Let $x_1, x_2 \in (0, 1)$, and suppose $f(x_1) = f(x_2)$. Then

$$\begin{aligned} \left(x_1 - \frac{1}{2}\right)\pi &= \left(x_2 - \frac{1}{2}\right)\pi \\ x_1 - \frac{1}{2} &= x_2 - \frac{1}{2} \\ x_1 &= x_2 \end{aligned}$$

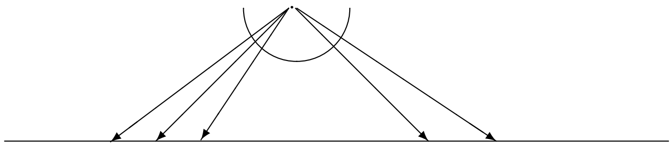
Thus f is injective. f is also surjective, since if $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ then

$$f\left(\frac{y}{\pi} + \frac{1}{2}\right) = \left(\left(\frac{y}{\pi} + \frac{1}{2}\right) - \frac{1}{2}\right)\pi = \left(\frac{y}{\pi}\right)\pi = y$$

Hence f is bijective. Then since $(0, 1)$ is uncountably infinite, $(-\frac{\pi}{2}, \frac{\pi}{2})$ is also uncountably infinite.

Since $(-\frac{\pi}{2}, \frac{\pi}{2})$ is uncountably infinite, and $h : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is bijective, it follows that $\mathbb{R}$ is also uncountably infinite. It can be shown that the cardinality of the set of real numbers $\mathbb{R}$ equals to the power set of the natural numbers $\mathbb{N}$, that is $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$. In fact, $|2^A| > |A|$ for every A. Here we see that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$, and so $\mathbb{R}$ must be uncountably infinite.

Below is the another way to see that the open interval $(0, 1)$ is equinumerous to $\mathbb{R}$. We take the interval $(0, 1)$ and bend it to form a semi circle. Then we can pair every point in $(0, 1)$ with its projection on the real line. Hence $\mathbb{R}$ is uncountably infinite.



$(0, 1) \approx \mathbb{R}$

Transcendental Numbers

We have shown that the set of algebraic numbers $\mathbb{A}$ is countably infinite, i. e. $|\mathbb{A}| = \aleph_0$. On the other hand, the set of real numbers $\mathbb{R}$ is uncountably infinite. Thus, there must exist other numbers. These numbers are called transcendental numbers, and they are defined as real non-algebraic numbers. Since $\mathbb{R}$ is uncountable, there are infinitely many transcendental numbers, i. e. $\mathbb{T}$ is uncountably infinite, and $|\mathbb{T}| = 2^{\aleph_0}$. Although there are infinitely many transcendental numbers, only a few of them are known. In 1844 Joseph Liouville proved the existence of transcendental numbers. And in 1851, Liouville constructed a transcendental number, known as Liouville's constant (Havil, 183). Thirty years later, in 1874, George Cantor in his paper published in $Crelle's\ Journal$ proved that there are infinitely many transcendental numbers.

Liouville's Theorem is stated as follows: "Let α be an irrational algebraic number of degree d. Then there exists a positive constant depending only on α, $c = c(\alpha)$, such that for every rational number $\frac{p}{q}$, the inequality

$$\frac{c}{q^d} \leq \left|\alpha - \frac{p}{q}\right|$$

is satisfied" (Burger and Tubbs, 11).

Liouville constant, the first number proved to be transcendental, is defined as follows:

$$\mathcal{L} = \sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001000 0...$$

This number is transcendental since it violates the inequality from Liouville's Theorem.

Among the most commonly used transcendental numbers, are e and π. Euler first proved in 1744 that e is irrational, and Liouville showed in 1840 that neither e nor e^2 could be rational or a quadratic irrational. But only in 1873 Hermite proved that e is a transcendental number. The work of Hermite was simplified by Hilbert, Hurwitz, and Gordan in 1893 (Baker, 3).

Below is the proof that e is transcendental. "First notice that if $f(x)$ is a real polynomial of degree m, and if

$$I(t) = \int_0^t e^{t-u} f(u)\, du$$

where t is an arbitrary complex number, then by integrating by parts we obtain

$$I(t) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t). \qquad (1)$$

Further, replacing coefficients with their absolute values, we obtain

$$|I(t)| \leq \int_0^t \left|e^{t-u} f(u)\right|\, du \leq |t|\, e^{|t|} \bar{f}(|t|) \qquad (2)$$

where $\bar{f}(x)$ is the polynomial obtained from $f(s)$. Now suppose that e is algebraic, such that

$$a_n e^n + a_{n-1} e^{n-1} + ... + a_1 e^1 + a_0 \qquad (3)$$

where the coefficients $a_0, a_1, ..., a_n$ are integers, and $a_n \neq 0$, $n \geq 1$ Then we can compare estimates for

$$J = a_n I(n) + a_{n-1} I(n-1) + ... + a_1 I(1) + a_0 I(0),$$

with

$$f(x) = (x-n)^p ... (x-1)^p x^{p-1},$$

where p is a large prime number. Then from equation (1) and (3) we get

$$J = -\sum_{j=1}^{m}\sum_{k=0}^{n} a_k f^{(j)}(k),$$

where $m = (n+1)p-1$. Then we see that $f^{(j)}(k) = 0$ if $j < p, k > 0$ and if $j < p-1, k = 0$; thus for all j, k other than $j = p-1, k = 0$, $f^{(j)}(k)$ is an integer divisible by $p!$. Further

$$f^{(p-1)}(0) = (p-1)!(-1)^{np}(n!)^p,$$

where, if $p > n$, $f^{(p-1)}(0)$ is an integer divisible by $(p-1)!$ but not by $p!$. It follows that, if also $p > |a_0|$, then J is a non-zero integer divisible by $(p-1)!$ and thus $|J| \geq (p-1)!$. But then the estimate $\bar{f}(k) \leq (2n)^m$ together with equation (2) gives us

$$|J| \leq |a_1 e \bar{f}(1) + ... + |a_n| n e^n \bar{f}(n) \leq c^p$$

for some c independent of p. But the estimates are inconsistent when p is sufficiently large. Thus we have proved that e is transcendental" (Baker, 4).

Transcendental nature of π was proved by a German mathematician Carl Lindemann in 1882.
He proved that "given any distinct algebraic numbers $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$, if $a_1 e^{\alpha_1} + a_2 e^{\alpha_2} + a_3 e^{\alpha_3} + ... + a_n e^{\alpha_n} = 0$ for algebraic numbers $a_1, a_2, a_3, ..., a_n$ then $a_1 = a_2 = a_3 = ... = a_n = 0$ (Havil, 200).

The Lidemann-Weierstrass Theorem states that e^a is transcendental for any non-zero algebraic number α (Havil, 200).

Using this theorem, we can show that π is transcendental. We know that $e^{i\pi}+1 = 0$ that is $e^{i\pi} = -1$. But -1 is algebraic, and so $i\pi$ can not be algebraic, thus it is transcendental. Further, we know that i is algebraic because it is a root of $x^2+1 = 0$ equation. Since the algebraic numbers are closed under addition and multiplication, if π were algebraic, then the product of π and i would be algebraic too. But this is a contradiction. Thus π is is transcendental.

Conclusion

In this paper we explored some ideas and concepts from Set Theory, such as set, subset, empty set, power set, and intersection and union of sets. Using these concepts, we were able to show that two sets are equal if there exits a bijective map between these two sets. Further we showed that the set equality can be also applied to infinite sets. We first proved that the set of natural numbers $\mathbb{N}$ is a countably infinite set, and then we showed that there exist bijective maps between $\mathbb{N}$ and $\mathbb{Z}$, and between $\mathbb{N}$ and $\mathbb{Q}$. We also proved that the set of algebraic numbers $\mathbb{A}$ is also countably infinite. On the other hand, we proved that these is no bijective map from the set of natural numbers $\mathbb{N}$ to the open interval $(0, 1)$, thus showing that $(0, 1)$ is uncountably infinite. By constructing bijective maps from $(0, 1)$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$, and from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$, we proved that the set of real numbers $\mathbb{R}$ is uncountably infinite. This last result shows that there must exist some real non-algebraic numbers. Such numbers are called transcendental numbers, and they include some widely used numbers such as e and π.

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