DECIDABILITY, TOPOLOGICAL SEMANTICS AND COMPLETENESS FOR S4

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Abstract. This Capstone expounds several key theorems of Alfred Tarski and J.C.C. McKinsey that connect Boolean algebra, point-set topology, and the system of modal logic S4. We augment Boolean algebras with a new operation satisfying certain properties, and then use those algebras to give a decision procedure for S4. This result is then extended to topology to find a decision procedure for topological equations built from the operations of closure, union, intersection, and complement. This correspondence between S4 and topological spaces also leads to several interesting results about S4; for instance, there are infinitely many distinct modal functions of a single variable in S4. We then show by way of a new notion of dissectible spaces that S4 is complete with respect to the Cantor space, and in a certain sense also complete with respect to Euclidean space. Though we are principally concerned with S4, we also provide a proof of decidability of S5 and remark on its relation to topological spaces.

Contents

1. Introduction

This paper's primary aim is to explicate the results of J.C.C. McKinsey and Alfred Tarski concerning the system of modal logic S4 and its connections to Boolean algebra and topology. In section 2, we introduce the notion of an S4-algebra. It is shown that we can determine whether an arbitrary sentence of S4 is provable by inspecting all finite S4-algebras with less than a certain number of elements. This constitutes a proof of the decidability of S4, as well as a proof of S4's completeness with respect to the collection of all finite S4-algebras. We also indicate how to obtain the corresponding result for S5. In section 3, we define topological spaces in terms of a closure function. We show that this characterization is equivalent to the more usual open set definition, and derive some other theorems of topology. In section 4, we define a notion of an S4-algebra over a topological space, and prove that every S4-algebra is isomorphic to a subalgebra of the S4-algebra over some topological space. Thus we have a correspondence between topological equations and sentences of S4 (that is, a topological semantics), and so we obtain a decision procedure for topological equations. In section 5, we show that every finite S4-algebra is isomorphic to a subalgebra of the S4-algebra over any totally disconnected, dissectible topological space, and is also isomorphic to the S4-algebra relative to some open element over any dissectible topological space. We therefore establish that S4 is complete with respect to the Cantor space, and in a certain sense complete with respect to Euclidean space. We conclude with some historical remarks in section 6.

2. A Decision Procedure for S4

Definition 1. [4, 1.1-1.6] We call $\mathcal{A} = (K, \times, -)$ a Boolean algebra when K is a set, $-$ is a unary operation, and $\times$ is a binary operation that satisfy the following:

.1 K contains at least two elements.
.2 K is closed under $-$, $\times$.
.3 $\times$ commutes. That is, for $a, b \in K$, $a \times b = b \times a$.
.4 $\times$ associates. That is, for $a, b, c \in K$, $a \times (b \times c) = (a \times b) \times c$.
.5 For $a, b \in K$, if there exists a $c \in K$ for which $a \times -b = c \times -c$, then $a \times b = a$.
.6 For $a, b, c \in K$, if $a \times b = a$, then $a \times -b = c \times -c$.

We then define:

.7 $0 := a \times -a$
.8 $1 := -0$
.9 $a + b := -(-a \times -b)$
.10 $a < b := a \times b = a$

We will make frequent use of some elementary facts about Boolean algebras, so we state a number of them now.

Theorem 1. *[4, B1-B20] In every Boolean algebra $\mathcal{A} = (K, -, \times)$, the following hold:*

.1	$a = - - a$
.2	*De Morgan's Laws:* $(-(-a \times -b)) = a + b$*;* $(-(-a + -b)) = a \times b$*;* $(-a \times -b) = -(a + b)$*;* $(-a + -b) = -(a \times b)$
.3	$a \times a = a$
.4	$a < b$ *and* $b < a$ *implies* $a = b$
.5	$a < b$ *and* $b < c$ *implies* $a < c$
.6	$-a + a = 1$
.7	$a + a = a$
.8	$-1 = 0$
.9	$0 + a = a$
.10	$a < 1$
.11	$(a \times b) < a$
.12	$a < (a + b)$
.13	$(a \times b) + c = (a + c) \times (b + c)$
.14	$(a + b) < c$ *iff* $a < c$ *and* $b < c$
.15	$a < b$ *and* $c < d$ *implies* $(a + c) < (b + d)$

Proof. We opt to prove only a subset of these. For .3, we observe that since $a \times -a = a \times -a$, we have $a \times a = a$ by Definition 1.5.

For .4, we notice that $a < b$ means $a = a \times b$, and $b < a$ means $b = b \times a$. By Definition 1.3, $a \times b = b \times a$, and so $a = b$.

For .5, suppose $a < b$ and $b < c$. Then $a = a \times b$ and $b = b \times c$. Substituting the latter into the former, we have $a = a \times (b \times c)$, which by associativity is equivalent to $a = (a \times b) \times c$. Again using the fact that $a = a \times b$, we have $a = a \times c$, and so $a < c$.

For .6, observe that $-a + a = -(- - a \times -a)$, which by definition is -0, which again by definition is 1.

For .11, we are to show that $a \times b = (a \times b) \times a$. But by associativity and commutativity, $(a \times b) \times a = (a \times a) \times b$, which by .3 equals $a \times b$. □

In discussing the systems S4 and S5 we will take the symbols $\sim, \wedge, \diamond$ as primitive, and will use $p, q, r, ...$ to refer to propositional variables. We consider the sentence $p \vee q$ as shorthand for $\sim (\sim p \wedge \sim q)$, the sentence $p \rightarrow q$ as shorthand for $\sim \diamond (p \wedge \sim q)$, and the sentence $p \equiv q$ as shorthand for $(p \rightarrow q) \wedge (q \rightarrow p)$.

Definition 2. ([5, 125-6, 493-501] We take $\underline{\text{S4}}$ to be the system axiomatized by:

.1	$p \wedge q \rightarrow q \wedge p$
.2	$p \wedge q \rightarrow p$
.3	$p \rightarrow p \wedge p$
.4	$(p \wedge q) \wedge r \rightarrow p \wedge (q \wedge r)$

.5 $p \to \sim (\sim p)$
.6 $[(p \to q) \wedge (q \to r)] \to (p \to r)$
.7 $[p \wedge (p \to q)] \to q$
.8 $\diamond \diamond p \equiv \diamond p$

And for S5, we add to the above:
.9 $p \to \sim \diamond \sim \diamond p$

The two systems share the same rules of inference, which are:

.10 Conjunction: If p and q have been shown, we may obtain $p \wedge q$.

.11 Modus ponens: If p and $p \to q$ have been shown, we may obtain q.

.12 Replacement: Two propositions which have been demonstrated to be equivalent may be substituted for each other. For example, since $\diamond\diamond p \equiv \diamond p$ is an axiom, we may replace as many instances of $\diamond p$ as we like in a sentence by $\diamond \diamond p$.

.13 Substitution: Any proposition may be substituted into the propositional variables of an other sentence. For example, if we had established that $\diamond p \to \sim \diamond \sim \diamond p$, then we may substitute $(q \wedge \diamond q)$ for p to obtain $\diamond(q \wedge \diamond q) \to \sim \diamond \sim \diamond(q \wedge \diamond q)$.

Definition 3. [4, 2.1-3.2] We may augment a Boolean algebra $\mathcal{A} = (K, -, \times)$ with a unary operation $*$ satisfying the following conditions:

.1 K is closed under $*$, i.e. if $a \in K$ then $*a \in K$.
.2 If $a \in K$, then $a < *a$.
.3 If $a, b \in K$, then $*(a + b) = *a + *b$.
.4 $*0 = 0$.

We may then ask whether the following also hold:

.5 If $a \in K$, then $*a = **a$.
.6 If $a \in K$, then $*a = 1$ unless $a = 0$.

If $*$ satisfies .1-.5 we call $\mathcal{A}$ an S4-algebra. If .6 is satisfied in place of .5 we call $\mathcal{A}$ an S5-algebra. In either case, we will write $\mathcal{A} = (K, -, \times, *)$.

From 3.6 and 3.4 one can derive 3.5. For $*1 < 1$ because $a < 1$ for all a, and $1 < *1$ by 3.3., so $*1 = 1$. Thus for $a \neq 0$, $**a = **1 = *1 = *a$. And for $a = 0$, $**a = *a$ by way of 3.4. Therefore every S5-algebra is an S4-algebra.

Definition 4. [4, 314-5] We say an algebra $\mathcal{A} = (K, -, \times, *)$ verifies a sentence α of S4 (or S5) if and only if every substitution of elements from K into the propositional variables of α, along with the operations $-, \times, *$ for $\sim, \wedge, \diamond$ respectively, yields a value of 1, where by 1 we mean the unity of the Boolean algebra. If not, i.e. if there are elements of K such that when substituted into α we obtain a value different from 1, we say $\mathcal{A}$ falsifies α.

It is worth pointing out that the Boolean algebra $\{0, 1\}$ verifies every theorem of the propositional calculus and no others; this is expressed in the familiar notion of truth tables. One way to think of the goal of this section is as an attempt to extend the idea of truth tables to the propositional calculus that has been enriched with the operator $\diamond$ and a new kind of implication.

Example 1. Consider the S4-algebra $\mathcal{A}_1 = (\{0, 1\}, -, \times, *)$ where $-, \times$ are defined in the ordinary way, and $*0 = 0$, $*1 = 1$. Then:

.1 As we would expect, $\mathcal{A}_1$ falsifies the sentence $p \to (p \to q)$. Rewriting the sentence with the operations of $\mathcal{A}_1$ gives $-*[p \times --*(p \times -q)]$. By our definition of $*$, we can simplify to $-[p \times (p \times -q)]$. Now set $p = 1$, $q = 0$, and we have $-[1 \times (1 \times -0)] = -(1 \times 1) = -1 = 0$.

.2 To check that $\mathcal{A}_1$ verifies the sentence $\sim p \vee \sim \diamond \sim p$, we rewrite it as $-p + - * -p$, and then again as $-(- - p \times - - * - p) = -(p \times * - p)$ to convert to the basic operations. Now we observe that $-(1 \times * - 1) = -(1 \times *0) = -(1 \times 0) = -0 = 1$, and $-(0 \times * - 0) = -0 = 1$. Note that $\sim p \vee \sim \diamond \sim p$ is not a theorem of either S4 or S5 ([2], 24).

Example 2. Consider the S4-algebra $\mathcal{A}_2 = (\{0, a, b, 1\}, -, \times, *)$ where $a \times b = 0$, $*a = a$, $*b = 1$, $-a = b$,$-b = a$, and the rest of the operations are defined in the ordinary way. Then:

.1 $\mathcal{A}_2$ also falsifies $p \to (p \to q)$ on the same evaluation as in the preceding example.

.2 $\mathcal{A}_2$ falsifies the sentence $\sim p \vee \sim \diamond \sim p$ on the evaluation $p = a$, since $-(a \times * - a) = -(a \times 1) = -a = b \neq 1$.

.3 $\mathcal{A}_2$ fails to be an S5-algebra since $*a \neq 1$. As we might hope, $\mathcal{A}_2$ falsifies the sentence $\sim \diamond p \vee \sim \diamond \sim \diamond p$, which is sometimes given in the form $\diamond p \supset \Box \diamond p$ in an alternate axiomatization of S5. When translated into the operations of $\mathcal{A}_2$, we have $- * p + - * - * p = -(- - *p \times - - * - *p)$, which we can simplify to $-(*p \times * - *p)$. Substituting $p = a$, we have

$$-(*a \times * - *a) = -(a \times * - a)$$
$$= -(a \times *b) = -(a \times 1) = -a = b \neq 1.$$

.4 Likewise, $\mathcal{A}_2$ falsifies our S5 axiom $p \to \sim \diamond \sim \diamond p$ on the evaluation $p = a$. We see that $p \to \sim \diamond \sim \diamond p$ becomes $- * (a \times - - * - *a) = - * (a \times * - *a)$, which is $- * (a \times * - a) = - * (a \times *b)$. Then finally, we get $- * (a \times 1) = - * (a) = -a = b \neq 1$.

Theorem 2. *[6, Theorem 2] If a and b are elements of an S4-algebra, then:*

.1	If $a < b$ then $*a < *b$.
.2	If $*a = 0$, then $a = 0$.
.3	$a < b$ if and only if $(a \to b) = 1$.
.4	If $a < b_1$, $a < b_2$,..., $a < b_k$ then $a < b_1 \times b_2 \times ... \times b_k$.

Proof. For .1, suppose $a < b$. Then $b = a + b$, and so $*b = *(a + b) = *a + *b$, showing $*a < *b$.

For .2, if $*a = 0$ then $a < 0$, and so $a = a \times 0 = 0$.

For .3, if $a < b$, then $a \times -b = 0$, and hence $- * (a \times -b) = 1$ which is the same as $(a \to b) = 1$. On the other hand, if $(a \to b) = 1$, then $- * (a \times -b) = 1$, and so by .2, $a \times -b = 0$, from which it follows that $a < b$.

For .4, we see that $a < b_i$ means that $a = a \times b_i$. Then $\prod_{i=1}^{k} a = (a \times b_1) \times (a \times b_2) \times ... \times (a \times b_k) = (\prod_{i=1}^{k} a) \times (\prod_{i=1}^{k} b_i)$. But in a Boolean algebra, $\prod_{i=1}^{k} a = a$, so we have $a = a \times (b_1 \times b_2 \times ... \times b_k)$, or $a < b_1 \times b_2 \times ... \times b_k$. □

We are now in a position justify the names "S4-algebra" and "S5-algebra".

Theorem 3. *[6, Theorem 10] Every S4-algebra verifies every theorem of S4, and likewise for S5-algebras.*

Proof. Let $\mathcal{A} = (K, -, \times, *)$ be an S4-algebra. We are to show that $\mathcal{A}$ verifies the axioms of S4, and that application of the rules of inference for S4 preserve verification by $\mathcal{A}$. For the axiom 2.1, $p \wedge q \to q \wedge p$, we observe that substituting elements a, b from K yields $- * [(a \times b) \times -(b \times a)]$. From Boolean algebra, we obtain $- * [(a \times b) \times -(a \times b)]$, which equals $- * 0 = -0 = 1$.

For 2.2, we have $- * [(a \times b) \times -a)] = - * [(a \times -a) \times b] = - * (0 \times b) = - * 0 = 1$.

For 2.3, we have $- * [a \times -(a \times a)] = - * (a \times -a) = - * 0 = 1$.

For 2.4, we see that $- * [[(a \times b) \times c)] \times -[a \times (b \times c)]] = - * [[(a \times b) \times c)] \times -[(a \times b) \times c)]] = - * 0 = 1$.

For 2.5, we have $- * (a \times - - -a) = - * (a \times -a) = - * 0 = 1$.

For 2.6, first notice that $a \times -c < [(a \times -b) + (b \times -c)]$. Then by Theorem 2.1, $*(a \times -c) < *[(a \times -b) + (b \times -c)]$, and so $*(a \times -c) < *(a \times -b) + *(b \times -c)$. Then we see that $-[*(a \times -b) + *(b \times -c)] < - * (a \times -c)$. Then we rewrite the left hand side to obtain $- * (a \times -b) \times - * (b \times -c) < - * (a \times -c)$, i.e. $(a \to b) \times (b \to c) < (a \to c)$, and so by Theorem 2.3, $[(a \to b) \times (b \to c)] \to (a \to c) = 1$ as desired.

For 2.7, observe that $-b - a + (a \times -b)$. Since $(a \times -b) < *(a \times -b)$, it follows that $-b < -a + *(a \times -b)$. Then $-[-a + *(a \times -b)] < b$. But $-[-a + *(a \times -b)] = a \times - * (a \times -b) = a \times (a \to b)$. Again by Theorem 2.3, we have $[a \times (a \to b)] \to b = 1$.

For the modal axiom $\diamond\diamond p \equiv \diamond p$, substituting a from K yields $[-*(**a \times -*a)] \times [-* (*a \times -**a)]$. Simplifying with $**a = *a$, we obtain $[-*(*a \times -*a)] \times [-*(*a \times -*a)]$, which becomes $- * 0 \times - * 0 = 1 \times 1 = 1$.

For the rule of conjunction, if we suppose that α and β are each verified by $\mathcal{A}$, then $\alpha \wedge \beta$ will become 1×1 under every substitution, and so equal to 1.

For modus ponens, suppose that α and $\alpha \to \beta$ are each verified and that a and b are two arbitrary expressions of $\mathcal{A}$ resulting from a substitution into α and β respectively. Then $a \to b = - * (a \times -b) = - * (1 \times -b) = - * -b = 1$. So $* - b = 0$, which by Theorem 2.2 implies that $-b = 0$, and so $b = 1$.

For the rule of replacement, suppose $\alpha \equiv \beta$ is verified. Let a and b be the result of substituting elements from $\mathcal{A}$ into α and β respectively. Then we have $(a \to b) = 1$ and $(b \to a) = 1$. Then by Theorem 2.3, $a < b$ and $b < a$, which implies $a = b$. Then clearly substituting a for b into a third sentence will not change its value.

Finally, the rule of substitution holds by our definition of verification. For if α is verified, this means that every substitution of elements from $\mathcal{A}$ for the propositional variables of α yields a value of 1. So, we may substitute any element we like to obtain another verified sentence, which is precisely the rule.

For S5, we need only check that an S5-algebra $\mathcal{A}_1$ verifies the additional axiom 2.9. We see that $p \to \sim \diamond \sim \diamond p$ becomes $- * (a \times - - * - *a)$, which equals $- * (a \times * - *a)$. If $a = 0$, then we have $- * (0 \times * - *0) = - * 0 = -0 = 1$, and if $a \neq 0$, we have $- * (a \times * - 1) = - * (a \times *0) = - * (a \times 0) = - * 0 = -0 = 1$. $\square$

Notice that Examples 1 and 2 show that an S4-algebra may verify some sentences that are not theorems of S4. We shall call an S4-algebra that verifies only those theorems an <u>S4-characteristic algebra</u>, and define an S5-characteristic algebra analogously. We are ultimately going to show that such an algebra exists, but first we will take a short digression to show that no finite S4-algebra has this nice property. We also note that from Example 2.4 and the last theorem we have shown that axiom 2.9 is not redundant, and thus that S4 and S5 are indeed distinct systems.

Theorem 4. *[3, Theorem 1] No finite S4-algebra is S4-characteristic.*

Proof. Consider an S4-algebra $\mathcal{A}_1$ with strictly less than n elements. We will construct a sentence F_n such that $\mathcal{A}_1$ verifies F_n, and then show that F_n is not a theorem of S4. Let $F_n = \sum_{1 \leq i < k \leq n} (p_i \equiv p_k)$. That is,

$$F_n = (p_1 \equiv p_2) \vee (p_1 \equiv p_3) \vee ... \vee (p_1 \equiv p_n) \vee (p_2 \equiv p_3) \vee ... \vee (p_{n-1} \equiv p_n).$$

Where the p_i's are propositional variables. Since $\mathcal{A}_1$ has less than n elements, any evaluation of F_n will have some summand of the form $a \equiv a$, and so the whole sum will reduce to a statement that is verified.

Now, to see that F_n is not a theorem, we consider another S4-algebra $\mathcal{A}_2$. Let K_2 be the power set of the integers from 1 to n. Define $-$ as set complement and $\times$ as set intersection so that we have a Boolean algebra on K_2, and define the $*$ operation as $*\emptyset = \emptyset$ and $*a = \{1, ..., n\}$ otherwise. It is clear that $\mathcal{A}_2$ indeed satisfies the five conditions of Definition 3. Using this algebra we can find an evaluation such that each summand of F_n is false: Choose the p_i's so that each are distinct (we have enough elements to do this). Then $(p_i \equiv p_j) = - * (p_i \cap -p_j) \cap - * (p_j \cap -p_i)$. We want to show that this reduces to $\emptyset$, which amounts to showing that one of the terms of the outer product is non-empty, since then $*$ applied to that would yield $\{1, ..., n\}$ and so its complement would be $\emptyset$, making the whole product equal to $\emptyset$. From elementary set theory, we see that for non-empty sets, if $A \neq B$ then the complement of one must meet the other. Hence each $(p_i \equiv p_j)$ reduces to $\emptyset$, so the whole sum reduces to $\emptyset$. So we have shown that F_n is not satisfied by an S4-algebra, so it is not a theorem. $\square$

Clearly Theorem 4 also demonstrates that there is no finite S5-algebra.

Theorem 5. *[6, Theorem 11] There is an S4-characteristic algebra.*

Proof. Let Σ be the set of all sentences of S4. We cannot immediately make an algebra out of this set that will suffice because we will need a more useful notion of equality than identity of two sequences of symbols to show, for instance, that $*0 = 0$. To get around this difficulty, define a relation R on Σ as follows: for $\alpha, \beta \in \Sigma$, $\alpha R \beta$ iff $\alpha \equiv \beta$ is a theorem of S4. It is easy to see that R is a congruence relation. Reflexivity follows from the fact that $\alpha \equiv \alpha$ is a theorem of S4. For symmetry, if we suppose that $\alpha R \beta$, then $\vdash_{S4} \alpha \equiv \beta$. Then by the rule of replacement, we may replace the left hand side of the theorem $\alpha \equiv \alpha$ with β to obtain $\beta \equiv \alpha$, which shows that $\beta R \alpha$. In similar fashion we see that R is transitive and that R is compatible with the operations $\sim, \wedge, \diamond$.

We now define a new algebra $\mathcal{A}_C = (K, -, \times, *)$ where K consists of the equivalence classes of Σ under the relation R. For $[\alpha], [\beta] \in K$, define the operations in the natural way, i.e. $-[\alpha] = [\sim \alpha]$, $[\alpha] \times [\beta] = [\alpha \wedge \beta]$, and $*[\alpha] = [\diamond\alpha]$. Note that in $\mathcal{A}_C$, the element 1 is $[\top]$, the set of sentences equivalent to the truth constant. Hence $[1] = [p \vee \sim p]$. Similarly, the zero element is $[\perp] = [p \wedge \sim p]$.

We now show that $\mathcal{A}_C$ is an S4-algebra. The first condition of Definition 3 is obvious. For 3.2, we see that $[\alpha] < *[\alpha]$ means $[\alpha] < [\diamond\alpha]$. So by definition of $<$, we are to show that $[\alpha] = [\alpha] \times [\diamond\alpha] = [\alpha \wedge \diamond\alpha]$. But this merely asserts that $\vdash_{S4} (\alpha \equiv \alpha \wedge \diamond\alpha)$. Theorem 16.33 in [5] states $(p \to q) \equiv p \to (p \wedge q)$, and [5, Theorem 18.4] states $p \to \diamond p$ so we have the left to right implication. For right to left, we appeal to [5, Theorem 12.17] which states $p \wedge q \to q$.

Likewise, $\vdash_{S4} (\diamond\alpha \vee \diamond\beta) \equiv \diamond(\alpha \vee \beta)$ is exactly [5, Theorem 19.82], so 3.3 is satisfied. To see that $*[0] = [\diamond 0] = [0]$, notice that $0 \equiv p\wedge \sim p$, and we know that $\vdash_{S4} [(p\wedge \sim p) \equiv \diamond(p\wedge \sim p)]$ by [6, Theorem 7]. Finally, it is clear that 3.5 is satisfied as $\diamond p \equiv \diamond\diamond p$ is an axiom of S4.

To complete the proof, we show that any non-theorem of S4 is falsified by $\mathcal{A}_C$. Suppose α is not a theorem of S4. If p is a propositional variable in α, substitute $[p]$ for p. With this substitution we obtain $[\alpha]$. Suppose for a contradiction that $[\alpha] = [1]$. But this means $\vdash_{S4} \alpha \equiv \top$, which contradicts α being a non-theorem. $\square$

One can find an S5-characteristic algebra by way of a similar construction.

Since the set of propositional variables in S4 is countable, and sentences of S4 are built from finite applications of $\sim, \wedge$, and $\diamond$ to the propositional variables, the set Σ of sentences in S4 is countable as well. So $\mathcal{A}_C$ is clearly countable. One might ask, then, why we have not already found a decision procedure for S4, since it would appear that $\mathcal{A}_C$ supplies a negative test for theoremhood (we already have a positive test from the axioms and rules). That is, given a sentence α, we could attempt every evaluation on the basis of $\mathcal{A}_C$, and if α is not provable, we would eventually find a falsification. This cannot be done, however, because $\mathcal{A}_C$ is not itself decidable. Given two arbitrary elements $[\alpha], [\beta]$ of $\mathcal{A}_C$, we cannot tell whether $[\alpha] = [\beta]$, since all we know is that $[\alpha] = [\beta]$ if and only if $\vdash_{S4} \alpha \leftrightarrow \beta$. But deciding $\vdash_{S4} \alpha \leftrightarrow \beta$ is just the problem we are trying to solve. So in particular, we cannot decide whether an evaluation equals $[1]$ in $\mathcal{A}_C$.

Theorem 6. *[6, Theorem 12] Given an S4-algebra $\mathcal{A} = (K, -, \times, *)$ and a finite subset of K, $\{a_1, ..., a_n\}$, there is a finite S4-algebra $\mathcal{A}_F = (K_F, -_F, \times_F, *_F)$ such that*

.1 $a_i \in K_F$ *where* $i = 1, 2, ...n$.

.2 K_F *has at most* 2^{2^n} *elements.*

.3 *If* $-a \in K_F$ *then* $-_F a = -a$, *if* $(a \times b) \in K_F$ *then* $a \times b = a \times_F b$, *and if* $*a \in K_F$, *then* $*a = *_F a$.

Proof. Let K_F consist of every element obtained by a finite number of applications of the operations $-$ and $\times$ to the elements $a_1, ..., a_n$. In other words, let K_F be the Boolean subalgebra generated by those elements. Define $-_F$ and $\times_F$ by restricting them to K_F. From Boolean algebra, we see that K_F has at most 2^{2^n} elements.

Note that we could not just define $*_F$ in the same way, because we would risk introducing new elements and thus ruining our Boolean algebra. We need to define $*_F$ in such a way that it uses the elements already present.

So, define $*_F$ as follows. We say an element $b \in K_F$ covers an element $a \in K_F$ when $a < b$ and $*b \in K_F$. Let $*_F a$ be the product of $*$ applied to all the elements of K_F that cover a. That is, if the elements $b_1, ..., b_k$ are those elements of K_F that cover a, then $*_F a = *b_1 \times *b_2 \times ... \times *b_k$. We are to show that K_F equipped with $*_F$ satisfies the conditions of an S4-algebra. Since K_F is a subalgebra of K, it contains 0 and -0, which is 1. We know that $*1 = 1$, and $a < 1$ for every a, so $*_F a$

is always defined. And if $b_1, ..., b_k \in K_F$ cover a, then our definition of K_F ensures that their product is also in K_F. Hence K_F is closed under $*_F$.

We now show that if $a, *a \in K_F$ then $*a = *_F a$. Now, from the fact that $\mathcal{A}$ is an S4-algebra, $a < *a$. Hence a covers a. Then in the definition of $*_F a$, we will have $*_F a = *a \times *b_1 \times ... \times *b_k$ where the b_i's are the other elements (if any) that cover a. Hence from Boolean algebra we see that $*_F a < *a$. On the other hand, we observe that if a is covered by $b_1, ..., b_k$, then by Theorem 2.4 and Definition 3.2 we see $*a < *b_1 \times b_2 \times ... \times *b_k = *a_F$. Hence $*a = *_F a$, and so $\mathcal{A}_F$ meets the three conditions of the theorem.

To complete the proof, we show that $\mathcal{A}_F$ is an S4-algebra. Condition 3.1 of Definition 3 has already been done. and 3.2 follows from our definition of $*_F$ and Theorem 2.4. For 3.4, we observe that $0 \in K_F$ and since $*0 = 0$, $*0 \in K_F$ as well. Then by the preceding paragraph, $0 = *_F 0$. For 3.5, we are to show that $*_F a = *_F *_F a$. We already know that $*_F a < *_F *_F a$. For the other direction, suppose $b_1, ..., b_k$ cover a. Then $*b_1, ..., *b_k$ cover $*_F a$. Let $c_1, ..., c_l$ be the other elements covering $*_F a$. So we have $*_F *_F a = **b_1 \times ... \times **b_k \times *c_1 \times ... \times *c_l$. Since we are in a Boolean algebra, we we may multiply again by the $**b_i$'s to obtain

$$*_F *_F a = **b_1 \times ... \times **b_k \times *c_1 \times ... \times *c_l \times (**b_1 \times ... \times **b_k).$$

But $**b_i = *b_i$, and hence

$$*_F *_F a = **b_1 \times ... \times **b_k \times *c_1 \times ... \times *c_l \times (*b_1 \times ... \times *b_k),$$

which equals

$$*_F *_F a \times *_F a.$$

This shows that $*_F *_F a < *_F a$, and therefore $*_F *_F a = *_F a$.

Finally, we are to show 3.3, that $*_F$ distributes over addition in $\mathcal{A}_F$. So let $a, b \in K_F$ and let $a_1, ..., a_x$ cover a, let $b_1, ..., b_y$ cover b, and let $c_1, ...c_z$ cover $a + b$. Our goal is to show that $*_F(a + b) = *_F a + *_F b$. By the definition of $*_F$, we can rewrite this as $*c_1 \times ... \times *c_z = (*a_1 \times ... \times *a_x) + (*b_1 \times ... \times b_y)$. From Boolean algebra, we know that $(a \times b) + c = (a + c) \times (b + c)$. Repeated applications of this yields $*c_1 \times ... \times *c_z = (*a_1 + *b_1) \times (*a_1 + *b_2) \times ... \times (*a_x + *b_y)$, and since $\mathcal{A}$ is an S4-algebra, we can further simplify to $*c_1 \times ... \times *c_z = *(a_1 + b_1) \times ... \times *(a_x + b_y)$. Now let $*c_l$ be an element of the left hand side. Since $a + b < c_l$, it follows that $a < c_l$ and $b < c_l$. Hence c_l covers a and c_l covers b, which means that c_l is one of the a_i's and one of the b_j's, and so $*c_l$ is one of the elements of the right hand side. On the other hand, if $*(a_i + b_j)$ is an element of the right hand side, then a_i covers a and b_j covers b. Then $a_i + b_i$ covers $a + b$, which means that $*(a_i + b_j)$ is one of the elements of the left hand side, and so the equality holds. □

To extend Theorem 6 to S5, we need only show additionally that if we start with an S5-algebra, then $*_F a = 1$ when a is non-zero. If $b_1, ..., b_k$ cover a, then $*_F a = *b_1 \times ... \times *b_k = 1 \times ... \times 1 = 1$.

Theorem 7. *[6, Theorem 13] Let α be a sentence of S4 with just n subsentences and k propositional variables. Then α is a theorem if and only if it is verified by every S4-algebra with $\leq 2^{2^n}$ elements.*

Proof. If α is a theorem, then α is verified by every S4-algebra and in particular by every S4-algebra with $\leq 2^{2^n}$ elements. For the other direction, suppose that α is not a theorem. Then α is falsified by the S4-characteristic algebra $\mathcal{A}_C$, i.e. there are elements $b_1, ..., b_k$ such that when substituted into α we obtain something other than 1. Suppose the subsentences of α are $\alpha_1, ..., \alpha_n$, and that substituting $b_1, ..., b_k$ from $\mathcal{A}_C$ into α yields $a_1, ..., a_n$. We may suppose without loss of generality that a_n corresponds to α. These form a finite subset of $\mathcal{A}_C$, so we may let $\mathcal{A}_\alpha$ be the finite S4-algebra of the previous theorem. Notice that by including all of the subsentences under the substitution, we ensure that the modal elements that we need will be present in $\mathcal{A}_\alpha$. So, by the third condition of the previous theorem, $\mathcal{A}_\alpha$ and $\mathcal{A}_C$ agree on the elements $a_1, ..., a_n$. In particular, we see that $a_n \neq 1$ as an element of $\mathcal{A}_\alpha$, which means that $\mathcal{A}_\alpha$ falsifies α. $\square$

The last theorem requires no modification for S5.

Theorem 7 provides us, in theory, with a decision procedure for S4. Given a sentence α of S4 with n subsentences, we could construct all possible S4-algebras with less than or equal to 2^{2^n} elements. We could then inspect each one, trying every possible substitution into α. If every such S4-algebra verifies α, then α is a theorem, and of course, if one falsifies α, then α is not a theorem of S4. Of course, even a relatively simple sentence such as $\sim (\diamond \sim \diamond \sim p \wedge \sim \diamond p)$ (or in shorthand, $\diamond\Box p \supset \diamond p$) has nine subsentences. 2^{2^9} is a very large number, and we have not even begun to discuss how (or how long it takes) to actually construct all S4-algebras with $\leq 2^{2^9}$ elements. Thus one of the aims of the remaining sections is to recast this decision procedure in terms of more familiar objects than S4-algebras. But first we will show that we have have also obtained a decision procedure for equations of S4-algebras.

We will call an S4-algebraic <u>expression</u> any meaningful sequence of variables of an S4-algebra and the operations $-$, $\times$, and $*$. We will call an S4-algebraic <u>equation</u> any equation whose right and left hand sides are S4-algebraic expressions. We will say an S4-algebraic expression a <u>corresponds</u> to a sentence α when a results from replacing the symbols $\sim, \wedge, \diamond$ by $-, \times, *$ respectively and replacing the propositional variables of α with variables of the S4-algebra.

Lemma 1. *[6, 129] An S4-algebraic equation $a = b$ holds in an S4-algebra if and only if $(a + -b) \times (b + -a) = 1$.*

Proof. Suppose $a = b$ is true in some S4-algebra. Then $(a + -b) \times (b + -a) = (a + -a) \times (a + -a) = 1 \times 1 = 1$. For the other direction, suppose $(a + -b) \times (b + -a) = 1$. Then $a + -b = 1$ and $b + -a = 1$. The former equation implies $a < b$, and the latter implies $b < a$, so $a = b$. $\square$

Theorem 8. *[6, Theorem 17] Let $a = 1$ be an S4-algebraic equation. Then this equation holds in every S4-algebra if and only if the sentence α of S4 corresponding to a is a theorem.*

Proof. First, suppose that α is a theorem of S4. Then α is verified by every S4-algebra, which means precisely that $a = 1$ in every S4-algebra. For the other direction, suppose that α is not a theorem. Then by the method of Theorem 7 we can find a finite S4-algebra which falsifies α, i.e. one in which $a \neq 1$. $\square$

By Lemma 1 and Theorem 8, we have found a decision procedure for all S4-algebraic equations built from the operations of $-, \times, *$. We can also handle the relations $<$ and $=$ by the following theorem.

Theorem 9. *[6, Theorem 18] Let α and β be sentences of S4, and let a and b be the corresponding S4-algebraic expressions. Then $a < b$ in every S4-algebra if and only if $\alpha \to \beta$ is a theorem of S4, and $a = b$ if and only if $\alpha \equiv \beta$ is a theorem of S4.*

Proof. We know that $a < b$ if and only if $a \times -b = 0$. But this is equivalent to $*(a \times -b) = *0 = 0$, from which we obtain $- * (a \times -b) = 1$. By Theorem 9, this equation holds in every S4-algebra if and only if $\sim \diamond(\alpha \wedge \sim \beta)$ is a theorem of S4, which is equivalent to $\alpha \to \beta$ being a theorem. Similarly, $a = b$ if and only if $a < b$ and $b < a$. By what was just shown, these relations hold in every S4-algebra if and only if $\alpha \to \beta$ and $\beta \to \alpha$ are theorems, which is the same as saying that $\alpha \equiv \beta$ is a theorem. $\square$

3. Topological Spaces

Definition 5. [7, 145] A <u>topological space</u> is a set X equipped with a unary operation $\mathcal{C} : \mathscr{P}(X) \to \mathscr{P}(X)$ called the <u>closure</u> operation satisfying the following:

.1 If $A \subseteq X$, then $A \subseteq \mathcal{C}(A)$.
.2 If $A \subseteq X$, then $\mathcal{C}(A) = \mathcal{C}(\mathcal{C}(A))$.
.3 If $A, B \subseteq X$, then $\mathcal{C}(A \cup B) = \mathcal{C}(A) \cup \mathcal{C}(B)$.
.4 $\mathcal{C}(\emptyset) = \emptyset$.

We call a subset $A \subseteq X$ <u>closed</u> when $A = \mathcal{C}(A)$. If $\mathcal{C}$ satisfies, in addition to the above,

.5 If $A \subseteq X$ is a singleton, then $\mathcal{C}(A) = A$.

then we call X a <u>topological space in the narrow sense</u>.

Though this definition will be more convenient for our purposes, it is equivalent to the more usual way of defining a topology on X by way of specifying a collection of open sets.

Theorem 10. *[9, Theorem 3.7] Given a set X and a function $\mathcal{C} : \mathscr{P}(X) \to \mathscr{P}(X)$ satisfying conditions 5.1-4 and with closed sets defined in the manner described, we obtain a topology on X where the closure of a set A, defined as the intersection of all closed sets containing A, is exactly $\mathcal{C}(A)$.*

Proof. Let $\mathscr{F}$ be the collection of subsets $A \subseteq X$ for which $\mathcal{C}(A) = A$. We are to show that any intersection of members of $\mathscr{F}$ is still in $\mathscr{F}$, that finite union of members of $\mathscr{F}$ is still in $\mathscr{F}$, and that $X, \emptyset \in \mathscr{F}$. Then the topology τ on X will be defined as containing the complements of the members of $\mathscr{F}$.

First, observe that if $A \subseteq B$, then $B = A \cup (B \setminus A)$, and so $\mathcal{C}(B) = \mathcal{C}(A \cup (B \setminus A))$. Since $\mathcal{C}$ respects union, we have $\mathcal{C}(B) = \mathcal{C}(A) \cup \mathcal{C}(A \setminus B)$, it follows that $\mathcal{C}(A) \subseteq \mathcal{C}(B)$.

To show that arbitrary intersection of members of $\mathscr{F}$ remains in $\mathscr{F}$, suppose that for each i in some index set I we have $F_i \in \mathscr{F}$. Then $\cap_{i\in I} F_i \subseteq F_i$ for all i, and therefore $\mathcal{C}(\cap_{i\in I} F_i) \subseteq \mathcal{C}(F_i)$. Hence $\mathcal{C}(\cap_{i\in I} F_i) \subseteq \cap_{i\in I} \mathcal{C}(F_i) = \cap_{i\in I} F_i$. On the other hand, $\cap_{i\in I} F_i \subseteq \mathcal{C}(\cap_{i\in I} F_i)$ by condition 5.1, so we have $\cap_{i\in I} F_i = \mathcal{C}(\cap_{i\in I} F_i)$, which is what we wanted to show.

To show that finite union of members of $\mathscr{F}$ remains in $\mathscr{F}$, suppose that $F_1, ..., F_n \in \mathscr{F}$. Then by repeated applications of 5.3, $\mathcal{C}(F_1 \cup ... \cup F_n) = \mathcal{C}(F_1) \cup ... \cup \mathcal{C}(F_n)$. But for each i, $\mathcal{C}(F_i) = F_i$ by assumption, and so $F_1 \cup ... \cup F_n \in \mathscr{F}$.

$\emptyset \in \mathscr{F}$ follows immediately from 5.4, and $X \in \mathscr{F}$ follows from the fact that $X \subseteq \mathcal{C}(X)$ by 5.1, and $\mathcal{C}(X) \subseteq X$ since X is closed under the closure operation.

It remains to show that $\mathcal{C}(A)$ is the smallest closed set containing A. By 5.2, $\mathcal{C}(A)$ is a closed set, and by 5.1, $\mathcal{C}(A)$ does indeed contain A. On the other hand, if F is a closed set containing A, i.e. $A \subseteq F$, then $\mathcal{C}(A) \subseteq \mathcal{C}(F)$ by the fact noted above. But $\mathcal{C}(F) = F$ by the assumption that F is closed, so $\mathcal{C}(A) \subseteq F$, which completes the proof. □

As an aside, the following theorem shows that we could just as well have defined topological spaces in the usual way of open sets and reached the closure characterization that we want.

Theorem 11. *Given a set X and topology τ on X, define the closure of a set A as the intersection of all closed sets containing A. Then the closure induces a function $f : \mathscr{P}(X) \to \mathscr{P}(X)$ satisfying 5.1-4 and moreover,$\tau_f := \{X \setminus A : f(A) = A\}$ is equal to τ.*

Proof. Let $A \subseteq X$ and let $\{F_i\}_{i\in I}$ be the indexed family of all closed sets containing A. For 5.1, we see that since $A \subseteq F_i$ for each i, $A \subseteq \cap_{i\in I} F_i$. For 5.2, let $\{G_j\}_{j\in J}$ be the indexed family of closed sets containing $\cap_{i\in I} F_i$. Then $\cap_{i\in I} F_i \subseteq G_j$ for all j, and so $\cap_{i\in I} F_i \subseteq \cap_{j\in J} G_j$. Also, $\cap_{i\in I} F_i$ is itself a closed set containing $\cap_{i\in I} F_i$, so $\cap_{j\in J} G_j \subseteq \cap_{i\in I} F_i$.

For 5.3, we first note that if $A \subseteq B$ and $\{G_j\}_{j\in J}$ is the indexed family of closed sets containing B, then $\cap_{i\in I} F_i \subseteq \cap_{j\in J} G_j$, since each G_j is itself a closed set containing A and so is a member of the F_i's.

Now, let $B \subseteq X$, and let $\{H_k\}_{k\in K}$ be the family of closed sets containing $A \cup B$. Then we are to show that $(\cap_{i\in I} F_i) \cup (\cap_{j\in J} G_j) = \cap_{k\in K} H_k$. For the left to right inclusion, we see that since $A \subseteq A \cup B$, $\cap_{i\in I} F_i \subseteq \cap_{k\in K} H_k$ and likewise, $\cap_{j\in J} G_j \subseteq \cap_{k\in K} H_k$. For right to left, observe that $(\cap_{i\in I} F_i) \cup (\cap_{j\in J} G_j)$ is itself a closed set containing $A \cup B$, and so is a member of the H_k's.

Lastly, for 5.4 we observe that $\emptyset$ is a closed set containing $\emptyset$, so the intersection of all closed sets containing $\emptyset$ will equal $\emptyset$. So if we set $f(A)$ equal to the intersection of all closed sets containing A, then we have the function of the theorem statement.

For the second part of the theorem, let $A \in \{X \setminus A : f(A) = A\}$. So $f(-A) = -A$. In other words, if $\{F_i\}_{i\in I}$ is the collection of closed sets containing $-A$, then $-A = \cap_{i\in I} F_i$. But $\cap_{i\in I} F_i$ is a closed set, so $-A$ is closed which means $A \in \tau$. On the other hand, if $A \in \tau$, then $-A$ is closed and so $f(-A) = -A$ since the intersection of all closed sets containing $-A$ will be $-A$ itself. Then $A \in \tau_f$. □

Theorem 12. *Every topological space in the narrow sense is T_1. That is, if X is a topological space in the narrow sense, then for distinct points $x, y \in X$ there is a neighborhood of each not containing the other. Moreover, every T_1-space is a topological space in the narrow sense.*

Proof. Let X be a topological space in the narrow sense with closure operation $\mathcal{C}$, and let $x, y \in X$ be distinct points. Then $\mathcal{C}(\{x\}) = \{x\}$ and $\mathcal{C}(\{y\}) = \{y\}$, and so $-\mathcal{C}(\{x\})$ is an open set containing y but not x, and $-\mathcal{C}(\{y\})$ is an open set containing x but not y. For the other direction, suppose X is a T_1-space and $x \in X$. Then for each point different from x, there is an open set not containing x. Hence $X \setminus \{x\}$ is open, which implies that $\{x\}$ is closed. □

Lastly, we prove a lemma that will be helpful later on.

Lemma 2. *If X is a topological space, and A, B are subsets of X where A is open, then $A \cap \mathcal{C}(A \cap B) = A \cap \mathcal{C}(B)$.*

Proof. Since $A \cap B \subseteq B$, the left to right inclusion is clear. For right to left, suppose $a \in A \cap \mathcal{C}(B)$. Then we are to show that $a \in \mathcal{C}(A \cap B)$. Suppose not. Then since the complement of $\mathcal{C}(A \cap B)$ is open, there is an open set U containing a such that $U \cap \mathcal{C}(A \cap B) = \emptyset$, and furthermore, $U \cap (A \cap B) = \emptyset$. On the other hand, $U \cap A$ is also open and contains a. Since $a \in \mathcal{C}(B)$, we know that every neighborhood of a meets B. Hence $(U \cap A) \cap B \neq \emptyset$, which is a contradiction. □

Theorem 13. *[7, Theorem 2.1] If X is a topological space, then $\mathscr{P}(X)$ forms an S4-algebra under the operations of intersection, complement, and closure, and we will call this the S4-algebra over X. Any subset of $\mathscr{P}(X)$ closed under these operations does as well, and we call such a subset a subalgebra of the S4-algebra over X.*

Proof. This follows immediately from the definitions of topological spaces and S4-algebras, where we identify $*$ with closure, $-$ with set complement, and $\times$ with intersection. □

Example 3. [7, 148] Suppose X is a topological space, and call K the S4-algebra over X. Let K' be the collection of elements of K whose boundary is empty, i.e. $\mathcal{C}(x) \cap \mathcal{C}(-x) = \emptyset$. Then K' is a subalgebra of K. To see this, notice that if x has an empty boundary, then so does $-x$. And if $x, y \in K'$, then $-(x \cap y) = -x \cup -y$, so $\mathcal{C}(x \cap y) \cap \mathcal{C}(-(x \cap y)) = \mathcal{C}(x \cap y) \cap \mathcal{C}(-x \cup -y)$. Using the fact that closure respects union and then distributing, we obtain $[\mathcal{C}(x \cap y) \cap \mathcal{C}(-x)] \cup [\mathcal{C}(x \cap y) \cap \mathcal{C}(-y)]$. Since $x \cap y \subseteq x$ and $x \cap y \subseteq y$, it follows that each piece of this union is empty, and so K' is closed under intersection. Finally, since $\mathcal{C}(x) = \mathcal{C}(\mathcal{C}(x))$, we know that K' is closed under the closure operation.

Our next goal is to show the converse of Theorem 13, that given an S4-algebra we can find a topological space that contains a subalgebra isomorphic to it. But first, we take a short detour into Boolean algebras.

Definition 6. We define infinite sums in a Boolean algebra in the following way. Let $\mathcal{B}$ be a Boolean algebra and let A be an infinite set of elements of $\mathcal{B}$. Since $\leq$ is a partial order on $\mathcal{B}$, we may define $\cup_{x \in A} x$ as $supA$ if this supremum exists. (By $supA$ we mean an element a such that $x \leq a$ for all $x \in A$, and if $x \leq y$ for all $x \in A$, then $a \leq y$.)

Definition 7. Let A be a set of elements of a Boolean algebra $\mathcal{B} = (K, -, \times)$. If A is finite, then clearly the sum $\cup_{x \in A} x$ is well-defined. If this sum is defined for every countable set A, then we say $\mathcal{B}$ is <u>countably additive</u>. If it is defined for every set A then we say $\mathcal{B}$ is <u>completely additive</u>.

Notice that since we have defined addition in terms of multiplication, any countably (completely) additive Boolean algebra is also countably (completely) multiplicative.

Example 4. Consider the following Boolean algebras:

.1 Let K_1 be the collection of subsets of $\mathbb{R}$ that are either finite or have finite complement. Then K_1 is a Boolean algebra, but is not countably additive since every singleton containing an integer is a member of K_1, but the collection of all integers is not.

.2 Let K_2 be the collection of subsets of $\mathbb{R}$ that are either countable or finite or whose complement is countable or finite. K_2 is countably additive, but not completely additive since all singletons containing a non-negative real number are in K_2, but their union $[0, \infty)$ is not.

Lemma 3. *[7, Lemma 2.3] Given a completely additive Boolean algebra K with an operation $\mathcal{C}_S$ defined over a subset S of K satisfying the conditions of a closure operation, we can find an operation $\mathcal{C}$ defined on K such that K is an S4-algebra with respect to $\mathcal{C}$ and moreover, $\mathcal{C}_S$ and $\mathcal{C}$ agree for elements of S.*

Proof. Let K be a completely additive Boolean algebra with $\mathcal{C}_S$ a closure operation defined over the subset S of K. Similarly as in the proof of Theorem 6, we will say an element y of S covers an element x of K when $x < y$ and $\mathcal{C}_S(y) = y$. (Notice, however, that this is slightly different from the covering we used in Theorem 6.) We define $\mathcal{C}(x)$ on K as the product of all the elements that cover x. In the event that no element covers x, set $\mathcal{C}(x) = 1$.

First we are to show that $\mathcal{C}$ is indeed a closure operation. From the way we have defined covering, we see that $x < \mathcal{C}(x)$.

From this fact we also obtain $\mathcal{C}(x) < \mathcal{C}(\mathcal{C}(x))$. On the other hand, suppose that y covers x. Then since $\mathcal{C}(x)$ is a product of which y is one element, $\mathcal{C}(x) < y$ as well. Hence y covers $\mathcal{C}(x)$. So since every element that covers x also covers $\mathcal{C}(x)$, we will have $\mathcal{C}(\mathcal{C}(x))$ equal to some product that includes $\mathcal{C}(x)$, and so $\mathcal{C}(x) < \mathcal{C}(\mathcal{C}(x))$. From the two inclusions we see that we have shown $\mathcal{C}(x) = \mathcal{C}(\mathcal{C}(x))$.

Since $0 < 0$ and by definition, $\mathcal{C}_S(0) = 0$, we have $\mathcal{C}(0) = 0$.

Lastly, we are to show that $\mathcal{C}$ respects addition. We note that in a Boolean algebra the infinite distributive laws hold so long as the sums and products involved exist - and here they do by the assumption that K is completely additive and therefore also completely multiplicative. Now, suppose A_1, A_2, and A_3 are the sets of elements that cover x, y and $x + y$ respectively. We notice that every element of A_3 can be written in the form $z_1 + z_2$ where z_1 covers x and z_2 covers y. Then we observe that $\mathcal{C}(x) + \mathcal{C}(y) = \prod_{z_1 \in A_1} z_1 + \prod_{z_2 \in A_2} z_2$. By distribution, this can be rewritten as $\prod_{z_1 \in A_1} \prod_{z_2 \in A_2} (z_1 + z_2)$. But this is exactly $\prod_{z \in A_3} z$, which equals $\mathcal{C}(x + y)$ by definition.

To complete the proof, we show that if $x \in S$ then $\mathcal{C}_S(x) = \mathcal{C}(x)$. Since $x \in S$ implies $\mathcal{C}_S(x) \in K_S$, we see that $\mathcal{C}_S(x)$ covers x and so $\mathcal{C}_S(x) > \mathcal{C}(x)$. On the other hand, if y covers x, then by our definition of covering we have $x < y$ and $\mathcal{C}_S(y) = y$. This implies that $\mathcal{C}_S(x) < \mathcal{C}_S(\mathcal{C}_S(y)) = \mathcal{C}_S(y) = y$, i.e. $\mathcal{C}_S(x)$ is contained in every element covering x. Thus $\mathcal{C}_S(x) < \mathcal{C}(x)$, and so we have shown $\mathcal{C}(x) = \mathcal{C}_S(x)$. $\square$

Theorem 14. *[7, Theorem 2.4] Every S4-algebra is isomorphic with a subalgebra of the S4-algebra over a topological space.*

Proof. Let $\mathcal{A} = (K, -, \times, \mathcal{C})$ be an S4-algebra. By the Stone Representation Theorem, we know that there is some set X for which K is isomorphic to a collection of subsets $S_{\mathcal{A}}$ of X with the operations of set complement and intersection. We can make $S_{\mathcal{A}}$ an S4-algebra isomorphic to $\mathcal{A}$ in the natural way: if $a \in K$ is mapped to $A \in S$ under the isomorphism and $\mathcal{C}(A)$ is mapped to B, then set $\mathcal{C}_S(A) = B$. So to complete the proof it suffices to show that $S_{\mathcal{A}}$ is isomorphic to a subalgebra of the S4-algebra over some topological space. $\mathscr{P}(X)$ is a completely additive Boolean algebra, and $S_{\mathcal{A}}$ with the closure operation $\mathcal{C}_S$ satisfies the hypothesis of the previous theorem. Hence we can define a closure operation $\mathcal{C}$ for X so that X becomes a topological space, and then we will have $S_{\mathcal{A}}$ a subalgebra of the S4-algebra over this space. Since our original algebra $\mathcal{A}$ isomorphic to $S_{\mathcal{A}}$, we are done. $\square$

From this last theorem we see that an equation is true in every S4-algebra if and only if it is true in every topological space, and that therefore a topological equation built from the operations of set complement, intersection, closure, and subset is true if and only if the corresponding sentence of S4 is provable. Hence we obtain "for free" a number of interesting results about the system S4.

From Kuratowski's Closure-Complement theorem, we see that there are fourteen distinct modalities (by which we mean finite sequences of $\diamond$ and $\sim$) in S4, with implication relations between them analogous to the subset relations between the fourteen of topology. So, for example, we see that in S4,

$$\Box p \to \Box \diamond \Box p \to \diamond \Box p \to \diamond \Box \diamond p \to \diamond p$$

just as

$$\mathring{A} \subset \mathring{\overline{\mathring{A}}} \subset \overline{\mathring{A}} \subset \overline{\mathring{\overline{A}}} \subset \overline{A}$$

holds in all topological spaces, where $\Box$ is defined as $\sim \diamond \sim$, $\overline{A}$ is shorthand for $\mathcal{C}(A)$, and $\mathring{A}$ denotes the interior of A or equivalently, $-\mathcal{C}(-A)$.

It also follows that there are an infinite number of distinct modal functions of one variable, i.e. expressions built from one propositional variable and the operations $\sim, \wedge, \diamond$, due to the corresponding result in topology that if we set $\phi(A) = A \cap \mathcal{C}(\mathcal{C}(A) \cap -A)$, then $\phi(A)$ is distinct from $\phi(\phi(A))$ which is distinct from $\phi(\phi(\phi(A)))$ and so on. This confirms that in the proof of Theorem 6 we could not simply identify $*_F$ with $*$, since the set generated by applications of $*, \times$, and $-$ could be infinite ([6], 132-3).

Next, we obtain a decision procedure for topological equations independent from S4-algebras. Notice that just as in Lemma 1, we can use equations of the form $A = X$ to decide whether any topological equation holds in every topological space.

Theorem 15. *[6, Theorem 19] Let A be a topological expression containing n subexpressions. Then $A = X$ in every topological space if and only if $A = X$ in every topological space with $\leq 2^n$ points.*

Proof. If $A = X$ holds in every topological space then it does so in every topological space with $\leq 2^n$ points. For the other direction, if $A = X$ is not true in every topological space, then the corresponding sentence α is not a theorem of S4 and has n subsentences. Then there is an S4-algebra $\mathcal{A}$ with $\leq 2^{2^n}$ elements that falsifies α, say by the substitution of elements $a_1, ..., a_k$. By the method of Theorem 14, we can find a topological space X such that $\mathcal{A}$ is isomorphic to the S4-algebra over X; in fact, the element 1 from $\mathcal{A}$ will be mapped to the set X. Then substituting the elements corresponding to $a_1, ..., a_k$ under this isomorphism for the set variables of A will yield a value different from X, so we have found the topological space of the theorem statement. And since $\mathcal{A}$ had $\leq 2^{2^n}$ elements, we see that X has $\leq 2^n$ points. □

Our next goal is to extend these results to T_1-spaces.

Theorem 16. *[7, Theorem 2.5] Given a topological space X with closure operation $\mathcal{C}_X$, there exists a T_1-space Y such that the S4-algebra over X is isomorphic to a subalgebra of the S4-algebra over Y.*

Proof. Let h be a function with domain $\mathscr{P}(X)$. For a singleton x, let $h(x)$ be an infinite set so that for $x, y \in X$ and $x \neq y$, $h(x) \cap h(y) = \emptyset$. If A is a subset of X, define $h(A)$ as $\cup_{x \in A} h(x)$, and set $h(\emptyset) = \emptyset$. We then set $Y = h(X)$, and notice that clearly $h(A) \cup h(B) = h(A \cup B)$.

Now, define a new function $g : \mathscr{P}(Y) \to \mathscr{P}(X)$, where for $A \in \mathscr{P}(Y)$, $g(A)$ is the set containing those of points $x \in X$ for which $A \cap h(x)$ is infinite. Then we see that $g(A \cup B)$, being the set of points $x \in X$ such that $(A \cup B) \cap h(x)$ is infinite, is equal to the set of points x for which $(A \cap h(x)) \cup (B \cap h(y))$ is infinite. But this means that either $A \cap h(x)$ or $B \cap h(x)$ must be infinite, and so $g(A \cup B) = g(A) \cup g(B)$.

We also observe that if A is finite, then $g(A) = \emptyset$ since no set can intersect a finite set to produce an infinite one.

And lastly, notice that for $A \subseteq X$, $h(A) = \cup_{x \in A} h(x)$, and $g(\cup_{x \in A} h(x)) = \{y \in X : h(y) \cap \cup_{x \in A} h(x) \,\text{infinite}\}$. For $h(y) \cap \cup_{x \in A} h(x)$ to be infinite, it must be the case that $y \in A$ because otherwise $h(y) \cap h(x) = \emptyset$ for each $x \in A$. Moreover, for any $y \in A$, $h(y) \cap \cup_{x \in A} h(x) = \cup_{x \in A} h(x)$ since $h(y)$ will be one of the elements of the union. Therefore, $g(h(A)) = A$.

Now, define the closure operation $\mathcal{C}_Y$ on Y by $\mathcal{C}_Y(A) = A \cup h\mathcal{C}_X(g(A))$ for $A \subseteq Y$. Let's check that Y is indeed a T_1-space with respect to C_Y.

From the definition of C_Y, it is clear that $A \subseteq C_Y(A)$.

We have $\mathcal{C}_Y(A \cup B) = A \cup B \cup h\mathcal{C}_X g(A \cup B)$. Since g respects union, this is equal to $A \cup B \cup h\mathcal{C}_X(g(A) \cup g(B))$, and since $\mathcal{C}_X$ does as well, we have $A \cup B \cup h(\mathcal{C}_X g(A) \cup \mathcal{C}_X g(B))$. We already showed that h respects union too, so we obtain $A \cup B \cup h\mathcal{C}_X g(A) \cup h\mathcal{C}_X g(B)$, which we can rewrite as $[A \cup h\mathcal{C}_X g(A)] \cup [B \cup h\mathcal{C}_X g(B)]$, which equals $\mathcal{C}_Y(A) \cup \mathcal{C}_Y(B)$ by definition.

Now, $\mathcal{C}_Y\mathcal{C}_Y(A)$ $=\mathcal{C}_Y(A \cup h\mathcal{C}_X g(A))$, which by the previous paragraph equals $\mathcal{C}_Y(A) \cup \mathcal{C}_Y h\mathcal{C}_X g(A)$. Expanding $\mathcal{C}_Y$, this becomes $(A \cup h\mathcal{C}_X g(A)) \cup (h\mathcal{C}_X g(A) \cup h\mathcal{C}_X gh\mathcal{C}_X g(A))$. Applying $gh(A) = A$ and $\mathcal{C}_X\mathcal{C}_X(A) = \mathcal{C}_X(A)$ yields $A \cup h\mathcal{C}_X g(A) \cup h\mathcal{C}_X g(A) \cup h\mathcal{C}_X g(A)$, which reduces to just $A \cup h\mathcal{C}_X g(A)$.

Finally, $\mathcal{C}_Y(\emptyset) = \emptyset \cap h\mathcal{C}_X g(\emptyset) = \emptyset$, and if A is finite, then $\mathcal{C}_Y(A) = A \cup h\mathcal{C}_X g(A)$ and since g applied to a finite set yields $\emptyset$, we have $A \cup h\mathcal{C}_X(\emptyset) = A \cup h(\emptyset) = A$.

To complete the proof, we claim that h is an isomorphism between the S4-algebra over X and a subalgebra of the S4-algebra over Y. From the way we defined h, it is clear that union, intersection, and complement are preserved and that h is injective, and we see that $\mathcal{C}_Y h(A) = h(A) \cup h\mathcal{C}_Y gh(A) = h(A) \cup h\mathcal{C}_Y(A) = h(A \cup \mathcal{C}_Y(A))$. Since $A \subseteq \mathcal{C}_Y(A)$, this equals $h\mathcal{C}(A)$, as desired. $\square$

Example 5. To illustrate the method of the last theorem, consider the topological space $X = \{a, b, c, d\}$ where $\mathcal{C}_X(A) = X$ for all $A \subseteq X$ other than $\emptyset$, and $\mathcal{C}(\emptyset) = \emptyset$. Define h as follows:

- $h(a) = [0, 1)$, the interval of the real line
- $h(b) = [1, 2)$
- $h(c) = [2, 3)$
- $h(d) = [3, 4)$
- $h(\emptyset) = \emptyset$

Then by the method of the theorem, we will set $Y = [0, 4)$ and define g as prescribed. For instance, if $A = (0, \frac{\sqrt{2}}{2}) \cup (\pi, 4)$, then $g(A) = \{a, d\}$, because $h(a \cup d) \cap A = [0, 1) \cup [3, 4) \cap A$ is infinite. Then for $\mathcal{C}_Y(A)$ we will have $A \cup h\mathcal{C}_X g(A) = A \cup h\mathcal{C}_X(\{a, d\}) = A \cup h(\{a, b, c, d\}) = A \cup Y = Y$. Indeed, $\mathcal{C}_Y$ will behave just as $\mathcal{C}_X$ when applied to an infinite set A, since $\mathcal{C}_X g(A)$ will always give $\{a, b, c, d\}$ in this case. But notice that for $A = \{\frac{1}{2}\}$, $g(A)$ will be empty, and so the term $h\mathcal{C}_X g(A)$ will contribute nothing to the union and we will have $\mathcal{C}_Y(A) = A$ as it should in a T_1-space.

Clearly the idea of this example can be readily extended to the case where X is countably infinite. In the case where X is uncountable, we need only exercise more care in choosing the assigned intervals.

By Theorem 16, we see that a topological equation is true in every topological space if and only if it is true in every T_1-space, since if an equation if true in every topological space, it is true in every T_1-space because T_1-spaces are topological spaces, and on the other hand, if it is false in some topological space X, then it will also be false in the T_1-space containing a copy of X from the last theorem ([6], Theorem 21).

5. Universal Algebras for S4 and S5

In this section we will need some new notions. Note that in light of the last section, we will now use $\cap$ and $\times$ interchangeably to refer to the multiplication operation in an S4-algebra, and likewise for $\mathcal{C}$ and $*$, and for 1 and X as the unity element.

Definition 8. [7, Definitions 1.9-11] An S4-algebra $\mathcal{A}$ is called connected if $\mathcal{C}(x) \cap \mathcal{C}(-x) = 0$ implies $x = 1$ or $x = 0$. $\mathcal{A}$ is called well-connected if $\mathcal{C}(x) \cap \mathcal{C}(y) = 0$ implies $x = 0$ or $y = 0$. $\mathcal{A}$ is called totally disconnected if every non-empty open element can be written as the sum of two disjoint non-empty open elements.

Our definition of connectedness is the ordinary one from topology. Well-connectedness may thought of as the property that all closed sets must meet. Since in an S5-algebra $\mathcal{C}(x) = 1$ for all non-zero x, every S5-algebra is well-connected. As far as examples, the S4-algebra over the space X of Example 5 is well-connected. For another, if we consider the sets A_1, A_2, A_3 where A_1 is the boundary of the unit circle, A_2 is the interior of the unit circle, and A_3 the exterior, then the S4-algebra consisting of these sets, the empty set, and their unions is well-connected, since the closure of each of these sets (except $\emptyset$) will include A_1 with the ordinary closure operation ([7], 147). The rational numbers with the usual topology form a totally disconnected space, since given an open connected set (a, b) we can divide this into two open sets by choosing an irrational c in the interval and partitioning (a, b) as $(a, c) \cup (c, b)$. Clearly we can extend this idea as needed if given an open set that is not connected. The Cantor set also totally disconnected.

Definition 9. [7, Definition 3.1] Given an S4-algebra $\Gamma = (K, -, \cap, \mathcal{C})$ and an element $a \in K$, $a \neq \emptyset$, we call $\Gamma_a = (K_a, -_a, \cap_a, \mathcal{C}_a)$ the relative algebra of Γ with respect to a. K_a consists of those elements $x \in K$ such that $x \subseteq a$. We identify $\cap_a$ with $\cap$, and define $-_a x := -x \cap a$ and $\mathcal{C}_a(x) = \mathcal{C}(x) \cap a$.

Notice that this notion of a relative algebra is carried over directly from topology.

Corollary 1. *[7, Corollary 3.2]* Γ_a *is an S4-algebra, and moreover, if a is open, the open elements of* Γ_a *are also open elements of* Γ.

Proof. From the way the operations of Γ_a are defined, it is clear that Γ_a is closed under $\mathcal{C}_a$.

For 3.2: If $x \in K_a$, $\mathcal{C}_a(x) = \mathcal{C}(x) \cap a$. Since $x \subseteq \mathcal{C}(x)$ and $x \subseteq a$, we know that $x \subseteq \mathcal{C}(x) \cap a$.

To see that $\mathcal{C}_a$ respects addition, notice that $\mathcal{C}_a(x \cup y) = (\mathcal{C}(x \cup y)) \cap a = (\mathcal{C}(x) \cup \mathcal{C}(y)) \cap a$, which we can rewrite as $(\mathcal{C}(x) \cap a) \cup (\mathcal{C}(x) \cap a) = \mathcal{C}_a(x) \cup \mathcal{C}_a(a)$.

For 3.4: Observe that $\mathcal{C}_a(\emptyset) = \mathcal{C}(\emptyset) \cap a = \emptyset \cap a = \emptyset$.

Finally, to see that $\mathcal{C}_a(\mathcal{C}_a(x)) = \mathcal{C}_a(x)$, i.e. $\mathcal{C}(\mathcal{C}(x) \cap a) \cap a = \mathcal{C}(x) \cap a$, we notice that $\mathcal{C}(x) \cap a \subseteq a$ and $\mathcal{C}(x) \cap a \subseteq \mathcal{C}(\mathcal{C}(x) \cap a)$, so the right to left inclusion holds. For left to right, observe that $\mathcal{C}(x) \cap a \subseteq \mathcal{C}(x)$, so $\mathcal{C}(\mathcal{C}(x) \cap a) \subseteq \mathcal{C}(\mathcal{C}(x)) = \mathcal{C}(x)$, and therefore $\mathcal{C}(\mathcal{C}(x) \cap a) \cap a \subseteq \mathcal{C}(x) \cap a$.

To prove the second part of the theorem, suppose a is open and $x \in \Gamma_a$ is open. This means that $-_a(\mathcal{C}_a(-_a x)) = x$. Simplifying the left hand side, we have $-_a(\mathcal{C}_a(-x \cap a)) = -_a(\mathcal{C}(-x \cap a) \cap a) = -[\mathcal{C}(-x \cap a) \cap a] \cap a$. By Lemma 2 (which we are justified in applying to S4-algebras by the results of the last section), we can simplify further to $-[\mathcal{C}(-x) \cap a] \cap a$, which becomes $(-\mathcal{C}(-x) \cup -a) \cap a$. Distributing, we obtain $(-\mathcal{C}(-x) \cap a) \cup (-a \cap a)$, and so we have $-\mathcal{C}(-x) \cap a = x$, showing that $-\mathcal{C}(-x) = x$, i.e. x is an open element of Γ. □

Notice, however, that the closed elements of Γ_a need not be closed in Γ. An easy counterexample is the relative algebra of the real line with respect to $(0, 1)$. There, $(0, \frac{1}{2}] = \mathcal{C}((0, \frac{1}{2}]) \cap (0, 1) = [0, \frac{1}{2}] \cap (0, 1) = (0, \frac{1}{2}] = \mathcal{C}_{(0,1)}((0, \frac{1}{2}])$ but $\mathcal{C}((0, \frac{1}{2}]) = [0, \frac{1}{2}]$. However, if x is closed in Γ and $x \subseteq a$, then x is closed in Γ_a, because from $x = \mathcal{C}(x)$ we see that $\mathcal{C}_a(x) = \mathcal{C}(x) \cap a = x \cap a = x$.

Definition 10. [7, Definition 3.3] We call an algebra Γ a universal algebra with respect to a set $\mathcal{U}$ of algebras if for each algebra $\mathcal{D} \in \mathcal{U}$, there is a subalgebra Δ of Γ such that Δ is isomorphic to $\mathcal{D}$. If instead we can find an open element a such that $\mathcal{D}$ is isomorphic to a subalgebra of Γ_a for each $\mathcal{D} \in \mathcal{U}$, we call Γ_a a generalized universal algebra with respect to $\mathcal{U}$.

Observe that if Γ is a universal algebra for $\mathcal{U}$, then Γ is also a generalized universal algebra for $\mathcal{U}$; we can see this by simply setting a equal to the unity element of Γ. In this section we are concerned primarily with finding a universal algebra for the collection of all finite S4-algebras. One may intuitively think of such a universal algebra as an object that is 'large enough' and possessing the right properties to contain every finite S4-algebra (i.e. an isomorphic copy of each). Since S4 is complete with respect to those algebras, S4 would be complete with respect to such an object.

Definition 11. [7, Definition 3.4] We call an S4-algebra Γ dissectible when for every non-empty non-empty element a and every pair of integers r, s with $r \geq 0$, $s > 0$, we can find $r + s$ non-empty pairwise disjoint elements $a_1, ..., a_r, b_1, ..., b_s$ from Γ such that:

.1 $a_1, ..., a_r$ are open

.2 The b_i's all have the same closure, i.e. $\mathcal{C}(b_1) = ... = \mathcal{C}(b_s)$

.3 The sum of these elements is a, i.e. $a_1 \cup ... \cup a_r \cup b_1 \cup ... \cup b_s = a$

.4. The boundary of a is contained in the closure of each b_i, which is in turn contained in the closure of each a_j. That is, $\mathcal{C}(a) \cap -a \subseteq \mathcal{C}(b_i) \subseteq \mathcal{C}(a_j)$.

We will call a collection of elements $a_1, ..., a_r, b_1, ..., b_s$ satisfying the above a dissection of a.

Although these conditions appear quite restrictive, and it takes considerable effort to show that an S4-algebra is indeed dissectible, we shall see shortly that the familiar S4-algebra over Euclidean space is dissectible. We also see from this definition and from Corollary 1 that if Γ is dissectible, and a is open, then Γ_a is dissectible.

For our next theorem we need to recall some more notions from topology. A topological space X is dense-in-itself when every point of X is a limit point of X, i.e. when there are no isolated points. We say X is normal (also called T_4) when for every pair of closed subsets A, B that do not meet, we can find two disjoint open sets C, D such that $A \subseteq C$ and $B \subseteq D$. We say that X has a countable basis (also called second-countable or completely separable) if there is a sequence $X_1, X_2, ..., X_n, ...$ of non-empty open subsets of X such that every non-empty open subset of X can be written as a union of elements from that sequence, i.e. when for $A \subseteq X$, $A \neq \emptyset$, there are indices $i_1, ..., i_n, ...$ such that $A = X_{i_1} \cup ... \cup X_{i_n} \cup ...$. Also note that a space with a countable basis must have a countable dense subset; we can find it by choosing an element from each member of the basis.

Theorem 17. *[7, Theorem 3.5] The S4-algebra over every normal, dense-in-itself topological space with a countable basis is dissectible.*

Proof. Let X be a normal, dense-in-itself topological space with a countable basis, let A be a non-empty open subset of X, and let $r, s \in \mathbb{Z}$ with $r \geq 0$ and $s > 0$.

We notice that X being normal and second countable implies that X is metrizable by Urysohn's metrization theorem. Moreover, we see that there is a countable subset $E = \{e_1, e_2, ...\}$ such that every point of X is a limit point of E.

From topology we will use the ordinary definition of the distance from a point to a set. That is, $d(x, B) = inf\{d(x, b) : b \in B\}$. We generalize this notion by defining the distance between two sets, $d(A, B)$, as $inf\{d(a, b) : a \in A, b \in B\}$, and say that if A or B is the empty set, then $d(A, B) = 1$. We also will use a new but straightforward notion: for a set A, define $m(A) = inf\{d(a, A \setminus \{a\} : a \in A\}$ where A is non-empty, and set $m(\emptyset) = 1$. To give an example from the real line, $m(\{17, 19, 23, 29\}) = 2$. Finally, we will denote the open ball of radius r with center a by $B(a, r)$ and the closed ball by $\overline{B}(a, r)$.

Now, to construct the sets $A_1, ..., A_r, B_1, ...B_s$ that will dissect A, we will define inductively two real numbers ε_n, δ_n, and sets $U_n, V_n, H_n^1, ..., H_n^r, K_n^1, ..., K_n^s$.

For our initial values, set $\varepsilon_0 = 1$, $\delta_0 = 1$, and each of the sets equal to $\emptyset$, except for U_0 which we set equal to A.

Now, suppose that the first n numbers and sets have been defined. Let u, v be the first two elements of E that land in U_n (we know that u, v exist since E must meet every non-empty set). Then define $\varepsilon_{n+1} = \frac{1}{3} min\{\frac{1}{n+1}, d(u, -U_n), d(u, v)\}$.

Let $x_1, ..., x_{r+s}$ be the first $r + s$ points from E that land in $B(u, \varepsilon_{n+1})$. (Notice that $u = x_1$.) Then define $\delta_{n+1} = \frac{1}{3} min\{d(\{x_1, ..., x_{r+s}\}, -B(u, \varepsilon_{n+1}), m(\{x_1, ..., x_{r+s}\})\}$.

We set $H_{n+1}^1 = B(x_1, \delta_{n+1})$,...,$H_{n+1}^r = B(x_r, \delta_{n+1})$, and set $K_{n+1}^1 = \{x_{r+1}\}$,...,$K_{n+1}^s = \{x_{r+s}\}$. From the way we have defined δ_{n+1} we see that these sets all are inside $B(u, \varepsilon_{n+1})$, and they do not meet. Moreover, from the $\frac{1}{3}$ factor in δ_{n+1}, we see that there is some 'space' between them. Clearly they are also all non-empty.

Finally, set $V_{n+1} = \mathcal{C}(H^1_{n+1}) \cup ... \cup \mathcal{C}(H^r_{n+1}) \cup K^1_{n+1} \cup ... \cup K^s_{n+1}$, and $U_{n+1} = U_n \setminus V_{n+1}$.

We note two facts. First, for points $x, y \in V_n$, $d(x,y) \leq \frac{1}{n}$ since $\varepsilon_n < \frac{1}{n}$, and the points of V_n lie within an open ball of radius ε_n. Second, from $U_0 = A$ and the definitions of ε_n and δ_n, we see that V_n does not meet the boundary of A.

Now, we define our sets $A_1, ..., A_r, B_1, ..., B_s$:

$$A_1 = H^1_1 \cup ... \cup H^1_n \cup ...$$

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$$A_r = H^r_1 \cup ... \cup H^r_n \cup ...$$

$$B_1 = K^1_1 \cup ... \cup K^1_n \cup ...$$

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$$B_{s-1} = K^{s-1}_1 \cup ... \cup K^{s-1}_n \cup ...$$

$$B_s = A \setminus (A_1 \cup ... \cup A_r \cup B_1 \cup ... \cup B_{s-1}).$$

Since the sets H^m_n, K^m_n are all non-empty and mutually exclusive, $A_1, ..., A_r, B_1, ...B_s$ are non-empty and mutually exclusive as well. Since the H^m_n's are open balls, we see that the A_i's are open. From the definition of B_s, it is clear that the union of these sets equals A.

To see why the remaining two conditions are satisfied, let us review our construction process. In the first iteration, we began with the whole set A. We then identified the first two elements, u and v, from E that landed in A. We drew a small ball $B(u, \varepsilon_1)$ around u that did not include v, and remained within A while leaving some space between $B(u, \varepsilon_1)$ and the boundary of A. Within that ball, we found the first $r+s$ points from E. For the first r such points, we drew a small ball around each one that remained within $B(u, \varepsilon_1)$, and defined their radius so that they did not meet and left some space between them. We gave to each A_i one of those balls. For the other s points, we gave one each to the B_i's. We then set up the next iteration by removing all of those open balls and points we assigned, and repeated the process.

We see then that because of the space we left between U_n and the boundary of A, that at any given iteration there will be a point from E closer to the boundary that has not yet been assigned. So eventually, we will reach that point and give to each A_i a nearby open ball and to each B_i a nearby point. Hence for every boundary point of A, within each of these sets we can find a sequence of points approaching that boundary point. Therefore, $\mathcal{C}(A) \cap -A \subseteq \mathcal{C}(B_i)$ and $\mathcal{C}(A) \cap -A \subseteq \mathcal{C}(A_j)$.

Moreover, at a given iteration and for a given point x assigned to a B_i, we left some space between x and the points and sets given to the other A_j's and B_k's. So we know that there are infinitely many points closer to x from E that had not yet been assigned. So eventually, we can be assured that we will give to each A_j and each B_k a point near x. Hence for every point in B_i, we can find a sequence of points from each A_j and each B_k converging to that point. Thus the closures of the B_i's are equal, and by a similar argument we can conclude that $\mathcal{C}(B_i) \subseteq \mathcal{C}(A_j)$. □

To further illustrate the method of the last theorem, we give the following example.

Example 6. Let us give two iterations of a dissection of the open interval $(0, 1)$ of $\mathbb{R}$ where $r = s = 2$. Following the proof, we set the following sets and values, taking E to be an enumeration of the rationals.

- $\varepsilon_0 = \delta_0 = 1$
- $H_0^1 = H_0^2 = K_0^1 = K_0^2 = \emptyset$
- $U_0 = (0, 1)$
- $u = \frac{1}{2}$, $v = \frac{1}{3}$. These are the first two rationals from our enumeration that land in U_0.
- $\varepsilon_1 = \frac{1}{18}$, so $B(u, \varepsilon_1) = (\frac{4}{9}, \frac{5}{9})$

Now, we take the first four rationals (we choose four because $r + s = 4$) from the enumeration that are in $(\frac{4}{9}, \frac{5}{9})$ and set $x_1 = \frac{1}{2}$, $x_2 = \frac{5}{11}$, $x_3 = \frac{6}{11}$, $x_4 = \frac{6}{13}$. We set $\delta_1 = \frac{1}{429}$ so that intervals centered at these points with radius δ_1 will not meet and will not leave $(\frac{4}{9}, \frac{5}{9})$. We set:

- $H_1^1 = B(\frac{1}{2}, \frac{1}{429})$
- $H_1^2 = B(\frac{5}{11}, \frac{1}{429})$
- $K_1^1 = \{\frac{6}{11}\}$
- $K_1^2 = \{\frac{6}{13}\}$

Now we set up for the next iteration by setting $V_1 = \overline{B}(\frac{1}{2}, \frac{1}{429}) \cup \overline{B}(\frac{5}{11}, \frac{1}{429}) \cup \{\frac{6}{11}\} \cup \{\frac{6}{13}\}$ and $U_1 = (0, 1) \setminus V_1$. So we have removed two small closed intervals and two points from $(0, 1)$, and now we repeat the process. This time $u = \frac{1}{3}$ and $v = \frac{2}{3}$ as these are the first two rationals in our list that are left in U_1. We set $\varepsilon_2 = \frac{1}{9}$, and then consider $\overline{B}(\frac{1}{3}, \frac{1}{9})$. The first four rationals in this interval are $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, and $\frac{3}{8}$. Now we will have $\delta_2 = \frac{1}{189}$ and we will again define four sets:

- $H_2^1 = B(\frac{1}{3}, \frac{1}{189})$
- $H_2^2 = B(\frac{2}{5}, \frac{1}{189})$
- $K_2^1 = \{\frac{3}{7}\}$
- $K_2^2 = \{\frac{3}{8}\}$

Again we will remove from $(0, 1)$ these four sets along with the four sets from above and repeat for new u and v. It is worth pointing out that ε_k is defined so that as k increases, ε_k approaches zero even though in this case we had $\varepsilon_1 < \varepsilon_2$, and likewise for δ_k. We continue inductively defining the sets H and K, and set A_1 equal to the union of the interiors of the H_i^1's and A_2 equal to the union of the interiors of the H_i^2's. So A_1 will be an infinite union of smaller and smaller intervals dispersed throughout $(0, 1)$. We likewise set B_1 equal to the union of the K_i^1's and B_2 equal to the union of the K_i^2's together with every point not included in A_1, A_2, or B_1. So B_1 and B_2 are sets of points dispersed throughout $(0, 1)$.

So at each iteration we have chosen the next rational point in $(0, 1)$ that has not yet been assigned to one of A_1, A_2, B_1, B_2, drawn a small interval around it, and within that interval we give a small interval to A_1, a small interval to A_2, and a point each to B_1 and B_2. Note that δ is defined so that we cannot give to A_1 an interval (a, b) and then to A_2 an interval (b, c). If we did, then we could not deal with the point b as clearly it could not be contained within an interval disjoint from those already assigned, and it could not be placed into one of the B_i's because we want the closure of the B_i's to be equal.

Now, why is this a valid dissection? The A_i's are unions of open intervals so they are open, and the definition of B_2 ensures that the union of these sets will equal $(0, 1)$. Furthermore the sets are clearly disjoint. It remains to be seen why the boundary of $(0, 1)$ is included in the closure of the B_i's and A_i's. At a given iteration we can be assured that at a later iteration we will go far enough out into the sequence of rationals to find an unassigned point that is closer to say, 0, than any of the previously assigned points. Then using the assignment method we give to each A_i and each B_i a nearby point. So it will turn out that each set contains a sequence converging to 0 and likewise for 1. Similarly we can see that the closure of each A_i will contain the closure of the B_i's (this relies on the above mentioned fact about how ε and δ are defined so that there will be enough space between the intervals assigned). Finally the closure of B_1 and of B_2 will be equal because for every rational not in the A_i's we will have given to B_1 and to B_2 either that point itself or infinitely many nearby points.

Theorem 18. *[7, Corollary 3.6] The S4-algebra over Euclidean space is dissectible.*

Proof. This follows immediately from the fact that Euclidean space is dense-in-itself, normal, and has a countable basis. $\square$

Theorem 19. *[7, Theorem 3.7] If Γ is a dissectible S4-algebra and a is a non-empty open element of Γ, and Φ is a finite well-connected S4-algebra, then there is a subalgebra Δ of Γ_a such that $\Delta \cong \Phi$.*

Proof. Our proof proceeds by induction on the number of atoms in Φ. If there is just one, then there is nothing to check. So, we assume that the theorem holds for every finite well-connected S4-algebra with less than p atoms and that Φ contains p atoms.

Recall that for an algebra to be well-connected, every closed element must meet. Hence there is an atom b_1 that is contained in every closed element of Φ. Suppose that $b_2, ..., b_k$ are the other atoms whose closure equals $\mathcal{C}(b_1)$, if there are any. So we have $\mathcal{C}(b_1) = \mathcal{C}(b_2) = ... = \mathcal{C}(b_k)$.

Observe that $b_1 \cup ... \cup b_k \subseteq \mathcal{C}(b_1 \cup ... \cup b_k)$, which equals $\mathcal{C}(b_1) \cup ... \cup \mathcal{C}(b_k)$. But these closures are equal, so their sum is just $\mathcal{C}(b_1)$. On the other hand, if x is an atom and $x \subseteq \mathcal{C}(b_1)$, then $\mathcal{C}(x) \subseteq \mathcal{C}(\mathcal{C}(b_1)) = \mathcal{C}(b_1)$. We have suppose that b_1 is contained in every closed element, so $b_1 \subseteq \mathcal{C}(x)$, and thus $\mathcal{C}(b_1) \subseteq \mathcal{C}(x)$, so $\mathcal{C}(x) = \mathcal{C}(b_1)$. This shows that x is one of the b_i's. Therefore, we have $b_1 \cup ... \cup b_k = \mathcal{C}(b_1)$.

Let $c_1, ..., c_q$ be the other atoms of Φ besides the b_i's. Notice that since b_1 exists for sure, $q < p$.

We select from these atoms a set $d_1, ..., d_n$. Let d_1 be the first atom in the list of c_i's whose closure does not contain as proper part the closure of any of the c_j's. That is, if $\mathcal{C}(c_j) \subseteq \mathcal{C}(d_1)$, then $\mathcal{C}(c_j) = \mathcal{C}(d_1)$. We let d_2 be the first atom of the c_i's whose closure also satisfies this property and has a closure different from $\mathcal{C}(d_1)$, and so on.

First observe that it is possible to select the d_i's in this way. For if we look at c_1 and find that $\mathcal{C}(c_i) = \mathcal{C}(c_1)$ or $\mathcal{C}(c_i) \nsubseteq \mathcal{C}(c_1)$ for each i, then we set $c_1 = d_1$. If, on the other hand, say $\mathcal{C}(c_2) \subsetneq \mathcal{C}(c_1)$, then we may look at c_2 and repeat. Since there are only q of the c_i's, in the worst case we will have $d_1 = c_q$ and no other d_j's. Furthermore, it is clear that for every c_i there is a d_j such that $d_j \subseteq \mathcal{C}(c_i)$.

Thus we may define the following elements. Let $e_i = \cup_{d_i \subseteq \mathcal{C}(c_j)} c_j$ for $i = 1, ..., n$. That is, let e_i be the sum of all those c_j's whose closure contains d_i. By the last remark of the previous paragraph, it is clear that $e_1 \cup ... \cup e_n = c_1 \cup ... \cup c_q$.

Additionally, we set $e_0 = b_1 \cup ... \cup b_k$. Then we will have $e_0 \cup ... \cup e_n = X$, as every atom is contained in the left hand side.

We shall now show that each of $e_1, ..., e_n$ is open. Notice that $-e_i = - \cup_{d_i \subseteq \mathcal{C}(c_j)} c_j = \cap_{d_i \subseteq \mathcal{C}(c_j)} (X - c_j)$. We can write this intersection as the sum of m atoms $a_1, ..., a_m$. Now, we will show that $\mathcal{C}(a_1 \cup ... \cup a_m) = a_1 \cup ... \cup a_m$. The right to left inclusion is obvious. For left to right, suppose for a contradiction that there is an atom x included in the left hand side but not the right. Then since $\mathcal{C}(a_1 \cup ... \cup a_m) = \mathcal{C}(a_1) \cup ... \cup \mathcal{C}(a_m)$, we see that $x \subseteq \mathcal{C}(a_l)$ for some l. Since x is not in $a_1 \cup ... \cup a_m$, $x \subseteq e_i$. Then by definition, $d_i \subseteq \mathcal{C}(x)$, and hence $d_i \subseteq \mathcal{C}(a_l)$. But a_l is an atom, so this means that $a_l \subseteq e_i$, which contradicts a_l being in $-e_i$.

Since Γ is dissectible, we can find pairwise disjoint non-empty elements $a_1, ..., a_n, f_1, ..., f_k$ that meet the conditions for a dissection. Set $a_0 = f_1 \cup ... \cup f_k$.

We see that $\mathcal{C}_\Gamma(f_1) = \mathcal{C}_\Gamma(f_1) \cup ... \cup \mathcal{C}_\Gamma(f_k)$ from the definition of dissection, and so $\mathcal{C}_\Gamma(f_1) = \mathcal{C}_\Gamma(f_2) = ... = \mathcal{C}_\Gamma(f_k) = \mathcal{C}_\Gamma(a_0)$.

We let Δ_0 be the subalgebra of Γ_{a_0} consisting of of all sums of the elements $\emptyset, f_1, ..., f_k$. Notice that though Δ_0 uses the same operations, it is not the same as Γ_{a_0}; rather it results from treating the elements $f_1, ...f_k$ as if they were atoms.

Now, define a function $h_0 : \Phi_{e_0} \to \Delta_0$ as follows. For $x = b_{i_1} \cup ... \cup b_{i_j}$, an element of Φ_{e_0} (the relative subalgebra with respect to e_0) set $h_0(x) = f_{i_1} \cup_{a_0} ... \cup_{a_0} f_{i_j}$, and set $h_0(\emptyset) = \emptyset$.

h_0 is an isomorphism. This is obvious so far as bijection and preservation of operations of complement, union, and intersection are concerned. To see that $h_0(\mathcal{C}_{e_0}(x)) = \mathcal{C}_{\Delta_0} h_0(x)$: Observe that for $x = \emptyset$ we have both sides equal to $\emptyset$. For x non-empty, we have $\mathcal{C}_{e_0}(x) = e_0 = b_1 \cup ... \cup b_k$, so $h_0(\mathcal{C}_{e_0}(x)) = f_1 \cup ... \cup f_k = a_0$. On the other hand, $\mathcal{C}_{\Delta_0}(y) = \mathcal{C}_\Gamma(y) \cap a_0$ for non-empty y. But we have seen that $\mathcal{C}_\Gamma(y) = a_0$ for any sum of the f_i's, so $\mathcal{C}_{\Delta_0}(h_0(x)) = a_0$ as well.

Now, observe that for each $i = 1, ..., n$, the relative algebra Φ_{e_i} is well-connected since every non-empty element contains d_i by our definition of e_i. Also, we have seen that the number of atoms in e_i is at most q, which is less than p and so we may use our induction hypothesis. Since a_i is open in Γ by the definition of dissection, there is a subalgebra $\Delta_i \leq \Gamma_{a_i}$ such that $\Delta_i \cong \Phi_{e_i}$. That is, there is an isomorphism h_i between Δ_i and Φ_{e_i}.

So, define the function $h : \Phi \to \Gamma_a$ by $h(x) = h_0(x \cap e_0) \cup ... \cup h_n(x \cap e_n)$. Setting Δ equal to the range of h, we have $\Phi \cong \Delta$. Hence Δ is an S4-algebra, and indeed it is a subalgebra of Γ_a, and so we have proven the theorem. $\square$

Setting a equal to the unity element X of Γ of this last theorem gives the result that every dissectible S4-algebra is a universal algebra for all finite well-connected S4-algebras.

Then we see that since every S5-algebra is a well-connected S4-algebra (since the closure of any non-empty element is equal to the unity element), that every dissectible S4-algebra serves as a universal algebra for the set of all finite well-connected S5-algebras. In particular, we see that Euclidean space is a universal algebra for S5.

Theorem 20. *[7, Theorem 3.8] Every totally disconnected dissectible S4-algebra is a universal algebra for the set of all finite S4-algebras.*

Proof. Let Γ be a totally disconnected dissectable S4-algebra, and let Φ be a finite S4-algebra. As in the previous theorem, we proceed by induction on the number of atoms in Φ. As before, there is nothing to check in the base case. So, suppose that the theorem holds whenever Φ has less than p atoms, and that Φ has exactly p atoms.

If Φ is well-connected, then we are done by the previous theorem. So assume that Φ is not well-connected.

Let $c_1, ..., c_q$ be the atoms of Φ, and select a set of atoms $d_1, ..., d_n$ as in the proof of the previous theorem, and again set $e_i = \cup_{d_i \subseteq \mathcal{C}c_j} c_j$. Then again, $e_1 \cup ... \cup e_n = c_1 \cup ... \cup_q = X$. And since Φ is not well-connected, there is some atom that is not contained in every closed element, and therefore not contained in the closure of every atom. Thus each e_i contains less than p atoms, and we may apply our induction hypothesis to each Φ_{e_i}.

Since Γ is totally disconnected, there are n non-empty pairwise disjoint open elements $a_1, ..., a_n$ in Γ whose sum is the universal element of Γ. Moreover, the relative algebras Γ_{a_i} are each totally disconnected. Therefore, by our induction hypothesis, there is a subalgebra $\Delta_i \leq \Gamma_{a_i}$ such that $\Phi_{e_i} \cong \Delta_i$ for each i. If h_i is the function establishing this isomorphism, we may set $h(x) = h_1(x \cap e_1) \cup ... \cup h_n(x \cap e_n)$, and we will have an isomorphism between Φ and a subalgebra $\Delta = \mathit{Range}(h)$ of Γ. $\square$

From this last theorem we see that the S4-algebra over every dense-in-itself, totally disconnected subspace of Euclidean space serves as a universal algebra for all finite S4-algebras. Of particular interest, we see that the Cantor set and the rationals can serve as such.

Lemma 4. *[7, Lemma 3.9] Let K be an S4-algebra, and let K^{WC} be the set of ordered pairs (x, y) where $x \in K$ and $y = X$ or $y = \emptyset$. Define the operations of complement, union, and intersection on K^{WC} in the natural way, and define $\mathcal{C}^{WC}((x, y)) = (\mathcal{C}(x), X)$ except in the case where $x = \emptyset$ and $y = \emptyset$, in which case set $\mathcal{C}^{WC}((\emptyset, \emptyset)) = (\emptyset, \emptyset)$. Then K^{WC} is a well-connected S4-algebra.*

Proof. We take the zero element of K^{WC} to be $(\emptyset, \emptyset)$ and the unity element to be (X, X). That K^{WC} is indeed an S4-algebra is obvious. To see that it is well-connected, notice that the element $(\emptyset, X)$ is contained within the closure of every non-empty element. $\square$

Lemma 5. *[7, Lemma 3.10] If K and K^{WC} are related as in the previous lemma, then $K \cong K^{WC}_{(X,\emptyset)}$. Moreover, $(X, \emptyset)$ is open.*

Proof. Define the function $h : K \to K^{WC}_{(X,\emptyset)}$ by $h(x) = (x, \emptyset)$. Clearly $h(x)$ is in fact an element of $K^{WC}_{(X,\emptyset)}$, and it is obvious that h is indeed an isomorphism. We see that since $\mathcal{C}^{WC}(\emptyset, X) = (\mathcal{C}(\emptyset), X) = (\emptyset, X)$, the element $(X, \emptyset)$ is open. $\square$

Lemma 6. *[7, Lemma 3.11] If K is dissectible, then K^{WC} is dissectible as well.*

Proof. We notice that except for $(\emptyset, \emptyset)$, the closed elements of K^{WC} are the elements of the form (x, X) where x is closed in K, for such an element, $\mathcal{C}^{WC}((x, X)) = (\mathcal{C}(x), X)) = (x, X)$. Therefore, except for (X, X), the non-empty open elements of K^{WC} are of the form $(-x, -X)$ where x is closed in K, i.e. the open elements are all the elements $(y, \emptyset)$ and (X, X) where y is open in K.

Then, given an open element $(a, \emptyset) \in K^{WC}$ and integers r, s with $r \geq 0$, $s > 0$, we can find a dissection of a by the elements $a_1, ..., a_r, b_1, ..., b_s$ from K. Then we see that the elements $(a_1, \emptyset), ..., (a_r, \emptyset), (b_1, \emptyset), ..., (b_s, \emptyset)$ will dissect $(a, \emptyset)$. In the case of the element (X, X), we can also use the corresponding elements from K and just set the s-th element equal to (b_s, X) so that the sum will be (X, X). □

Theorem 21. *[7, Theorem 3.12] Every dissectible S4-algebra is a generalized universal algebra for the set of all finite S4-algebras.*

Proof. Let K be a dissectible S4-algebra and let F be a finite S4-algebra, and let F^{WC} be the S4-algebra of the preceding lemmas. Then F^{WC} is a well-connected finite S4-algebra, so there is a subalgebra $\Delta \leq K$ such that $F^{WC} \cong \Delta$. Since $(X, \emptyset)$ is open in F^{WC}, and isomorphism respects closure and thus preserves open and closed sets, $(X, \emptyset)$ must correspond to an open element a in Δ. Thus we can restrict this isomorphism to $F^{WC}_{(X,\emptyset)}$, and we see that $F^{WC}_{(X,\emptyset)} \cong \Delta_a$ since intersection is preserved under the isomorphism. Moreover, we see that $\Delta_a \leq K_a$ and by Lemma 5, $F \cong \Delta_a$. So we have found an open element a for which F is isomorphic to a subalgebra of K_a, which proves the theorem. □

From this last theorem we see that the S4-algebra over Euclidean space is a generalized universal algebra for the set of all finite S4-algebras.

Finally, we notice that it is false that every dissectible S4-algebra is a universal algebra for all finite S4-algebras. From Lemmas 4 and 6 we see that there exists a well-connected dissectible S4-algebra. Since every subalgebra of a well-connected algebra will be well-connected as well, we could not find a subalgebra isomorphic to a finite S4-algebra that is not well-connected ([7], 160).

6. Historical Notes

Modal logic dates back at least as far as Aristotle. Medieval logicians studied it as well, but it became neglected until the late 19th-century. Then, the Scottish logician Hugh MacColl expressed dissatisfaction with the material implication $p \supset q \equiv \sim p \vee q$ on the basis of examples such as the following. Let p be the statement "He will persist in his extravagency" and let q be the statement "He will be ruined." Then according to M.I., $\sim (p \supset q) \equiv \sim (\sim p \vee q) \equiv p \wedge \sim q$, or "He will persist in his extravagency and he will not be ruined." But in ordinary language, the negation of p implies q would be "It is possible that he persists in his extravagency without being ruined." His point was that the material implication did not capture our ordinary meaning of implication. This failure can be seen even more clearly by considering the 'paradoxes of implication', which are the theorems of the propositional calculus $p \supset (q \supset p)$ and $\sim p \supset (p \supset q)$. In words, anything true is implied by anything at all, and anything false implies anything at all. From these one can prove that $(p \supset q) \vee (q \supset p)$. In words, given any two propositions, one implies the other or vice versa.

MacColl himself never axiomatized modal logic, though he did introduce symbols for necessity and possibility ([4], 213-216). But C.I. Lewis saw as well that M.I. differs greatly from our ordinary understanding of a statement like "q follows from p", and so was led to develop the logic of strict implication. He found that strict implication had its own 'paradoxes', such as $(\sim (q \rightarrow p) \rightarrow \diamond \sim p)$ (If there is a proposition that does not strictly imply p, then p is possibly false) and $\sim \diamond p \rightarrow (p \rightarrow q)$ (If p is self-contradictory - i.e. impossible - then p strictly implies any proposition whatever). Lewis said of them: "Unlike the corresponding paradoxes of material implication, these paradoxes of strict implication are inescapable consequences of logical principles which are in everyday use. They are paradoxical only in the sense of being commonly overlooked, because we seldom draw inferences from a self-contradictory proposition" ([5], 174-5).

The reader may have noticed that the axiomatizations of S4 and S5 used in this paper differ from the modern ones. Here, we defined every axiom in terms of strict implication, and used only the possibility operator. In contrast, the modern account augments the propositional calculus with the operators $\Box$ and $\diamond$ and leaves strict implication unmentioned. This shift was due to Gödel and his work connecting intuitionistic logic and S4. There, he axiomatized S4 by way of the now standard schemas **K** ($\Box(p \supset q) \supset (\Box p \supset \Box q)$), **T** ($\Box p \supset p$), and **4** ($\Box p \supset \Box\Box p$), together with the necessitation rule (from p conclude $\Box p$) and the set of theorems of propositional logic. In the view of Blackburn, Rijke, and Venema, this later axiomatization is preferable on several counts. First, it is easier to show the distinctness of systems axiomatized in this fashion. Second, they provide a more natural semantics, which makes it is easier to tell whether a given system includes all the axioms that we want it to. That is, it is more readily seen that it captures the kinds of reasoning in which we are interested. For instance, Lewis did not consider the important system **KT** (which is between S2 and S4 in terms of strength), while each of his systems S1, S2, and S3 have waned ([1], 38-48).

All this being said, however, it was in this earlier period that the connections between algebra, topology, and modal logic were developed. First, in 1938 Tang Tsao-Chen gave the interpretation of $\diamond$ as the topological operation of closure and showed that if a sentence of S4 is provable, then the corresponding topological equation holds in all spaces ([6], 129). McKinsey in 1941 proved the other direction of the correspondence, and so provided a decision procedure for topological equations. Three years later Tarski and McKinsey demonstrated the completeness of S4 with respect to the Cantor set and the real line. These discoveries still generate interest, as evidenced by a 2005 publication by Awodey and Kishida giving topological semantics to first order modal logic and a 2006 paper by Mints and Zhang on the

Tarski-McKinsey proof. Moreover, the algebraic results of Tarski and Jónsson in 1952 would eventually become recognized as essential to Kripke semantics ([1], 41).

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