Extracting Entanglement from a Four-Level Quantum System

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Abstract

The four-level system is the simplest system from which entanglement can be extracted and analyzed; as such, I will derive the appropriate observable operators that can extract entanglement from a four-level mixed state. The entanglement of a general four-level quantum system is an open problem in quantum information theory. Quantum entanglement is a counterintuitive phenomenon in quantum mechanics with no analog in classical mechanics; it allows for systems of multiple objects in which measurements on these objects are more strongly correlated than any possible classical system. These correlations hold even if the objects are separated by considerable distances. These properties make quantum entanglement

a crucial resource for developing quantum encryption and quantum computing technologies. Using the mathematics of representation theory and linear algebra, I will separate a general foul-level state into two entangled two-level states, from which I can analyze the extent of the entanglement between the two systems.

1 An Introduction to Quantum Entanglement

It is an assumption of classical mechanincs that any physical system with multiple degrees of freedom can be partitioned into multiple subsystems, and that measurements on one subsystem are independent of measurements on any of the other system. Though this assumption is intuitive, it does not hold in the realm of quantum mechanics, as evidenced in the phenomenon known as quantum entanglement. In quantum entanglement, properties of a subsystem cannot be described independantly of the other subsystems, which allows for measurements among subsystems that are more strongly correlated than any possible classical theory. Perhaps most baffling is the fact that entanglement can occur among objects separated by an arbitrary distance, which implies that the laws of quantum mechanics are nonlocal.

To illustrate this phenomenon, we consider two systems, called system A and system B, each consisting of a single spin-$\frac{1}{2}$ particle (say, an electron). Each of these particles may be in either a "spin up" or "spin down" state, which we will represent with the vectors $|0\rangle$and $|1\rangle$, respectively. We can choose to represent these spin states as standard basis vectors

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Notice that the spin state of any single particle can be represented as a linear combination of these two basis vectors: $|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$, with the restriction $|c_0|^2 + |c_1|^2 = 1$. Suppose that the state vector of A is $|\psi\rangle_A = \frac{1}{\sqrt{2}} |0\rangle_A + \frac{1}{\sqrt{2}} |1\rangle_A$ and the state vector of B is $|\phi\rangle_B = \frac{1}{\sqrt{2}} |0\rangle_B - \frac{1}{\sqrt{2}} |1\rangle_B$. Also suppose we assign Alice to be the observer of System A and Bob to be the observer of System B. Notice that each observer will observe either spin up or spin down with equal probability, independent of the other observer's measurement. Now if we consider the cumulative system A and B, then the state vector of the combined system is the outer product of the two vectors:

$$|\psi\phi\rangle_{AB} = |\psi\rangle_A \otimes |\phi\rangle_B = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2} |0\rangle_A \otimes |0\rangle_B - \frac{1}{2} |0\rangle_A \otimes |1\rangle_B + \frac{1}{2} |1\rangle_A \otimes |0\rangle_B - \frac{1}{2} |1\rangle_A \otimes |1\rangle_B$$

As expected, each outcome has equal probability of occurring.

A key attribute of the state vector $|\psi\phi\rangle_{AB}$, however, is the ability to express the vector as the outer product of the state vectors of the two subsystems, that is, $|\psi\phi\rangle_{AB} = |\psi\rangle_A \otimes |\phi\rangle_B$. Such states are called separable states. However, there exist states that cannot be factored via the tensor product. The most general state of the combined system of A and B is

$$c_{00} |0\rangle_A |0\rangle_B + c_{01} |0\rangle_A |1\rangle_B + c_{10} |1\rangle_A |0\rangle_B + c_{11} |1\rangle_A |1\rangle_B$$

With the restriction $\sum_{i,j} |c_{ij}|^2 = 1$ (We will supress the "$\otimes$" operator from now on). Therefore, a possible state is $|\psi\phi\rangle_{AB} = \frac{1}{\sqrt{2}} |0\rangle_A |1\rangle_B + \frac{1}{\sqrt{2}} |1\rangle_A |0\rangle_B$.

Observe that the two possible outcomes $|0\rangle_A |1\rangle_B$ and $|1\rangle_A |0\rangle_B$ occurr with equal probability.

Let's observe the outcome space more closely. If Alice measures the spin of particle A, then her measurement will yield either $|0\rangle_A$ or $|1\rangle_A$ (each with 50% probability), thus collapsing the state vector to either $|0\rangle_A |1\rangle_B$ or $|1\rangle_A |0\rangle_B$. However, if Bob subseqently measures the spin of particle B, the outcome of the measurement is known with certainty, according to the outcome of Alice's measurement. If Alice finds her particle to be in the spin up state, then the wavefunction of the system collapses to $|0\rangle_A |1\rangle_B$, and Bob will definitely find his particle to be spin down; alternately, if Alice finds particle A in the spin down state, Bob will find particle B in the spin up state. And so, remarkably, when Alice measures the state of her particle, she instantly knows the state of Bob's particle, even if Bob's particle is lightyears away. The ability to tailor correlated measurements such that a system appears entangled is a major topic in quantum information theory, and this paper will attempt to shed some light on this topic.

2 Pure and Mixed Quantum States

In quantum mechanics there exist two types of uncertainties. The first type of uncertainty is standard quantum uncertainty - the concept that properties of a system are not well-defined until a measurement is conducted. The second type of uncertainty pertains to ignorance about the actual state of the system. For example, an operator may randomly act on a system's state vector with some probability, transforming it into another state vector (such a state will emerge when the system is randomly prepared via different procedures). Therefore, the system may be in one state or the other. When the state is known, this is called a pure state, otherwise, if there is some uncertainty over what state the system is in, then state is called a pure state.

A pure state can be expressed as a ket vector such as $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ - note that only the uncertainty inherent in this state is whether a measurement of spin will yield spin up or spin down. A mixed state, however, can not be expressed as a single state vector, as it is a statistical ensemble of pure states. Instead of using ket vectors to represent quantum states, it is convenient to use a what is known as a density matrix.

Definition: A **density matrix** ρ is an $n \times n$ matrix with the following properties:

1. ρ is Hermitian, or self-adjoint. That is, $\rho^\dagger = \rho$, where $\rho^\dagger$ is the conjugate transpose of ρ.
2. Trace$[\rho] = 1$, where Trace$[\rho]$ is the sum of the diagonal entries of ρ.
3. ρ is positive semidefinite. Since ρ is Hermetian, this is equivalent to saying that all eigenvalues of ρ are nonnegative.

Density matrices prove to be a convenient way to express a mixed state. If the state of the system is a statistical ensemble of pure states $\{\psi_1, \psi_2, \ldots, \psi_n\}$ with probabilities $p_1, p_2, \ldots, p_n$. Then the density matrix of the mixed state is given by

$$\rho = \sum_{i=1}^{n} |\psi_1\rangle \langle\psi_1|$$

Thus the pure state $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ has a density matrix of

$$\rho_{pure} = |\psi\rangle \langle\psi| = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

If instead we have a mixed state that has a 50% chance of being in state $|\psi\rangle$

and a 50% chance of being in state $|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$, the the state has a density matrix:

$$\rho_{mixed} = \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|\phi\rangle\langle\phi| = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Note that ρ_{pure} and ρ_{mixed} both have a trace of 1, are self-adjoint, and a straightforward calculation shows that their eigenvalues are nonnegative: they are indeed density matrices.

Note that a density matrix is not unique to a particular statistical ensemble of states. For example, if we have a mixed state with a 50% chance of being in state $|\psi'\rangle = |0\rangle$ and a 50% chance of being in state $|\phi'\rangle = |1\rangle$, then the corresponding density matrix would be:

$$\rho'_{mixed} = \frac{1}{2}|\psi'\rangle\langle\psi'| + \frac{1}{2}|\phi'\rangle\langle\phi'| = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

and so $\rho_{mixed} = \rho'_{mixed}$

Now the trace of a density matrix is 1, regardless of whether the matrix represents a pure or mixed state. This is not case when when the density matrix is squared, and it provides us with a useful indicator of whether the matrix represents a pure or mixed state. For the trace of any pure state density matrix is 1, and the trace of any mixed state density matrix is less than 1. Observe that, for our density matrices above, $\text{Trace}\left[\rho^2_{pure}\right]=1$ and $\text{Trace}\left[\rho^2_{mixed}\right]=\frac{1}{2}$. Moreover, we can show that the trace of the square of a density matrix is bounded both above and below.

Theorem: If ρ is an $n \times n$ density matrix, then $\text{Trace}\left[\rho^2\right] \in \left[\frac{1}{n}, 1\right]$ for all $n \in \mathbb{N}$.

Proof: According to the spectral theorem for Hermetian matrices, $\rho =$

UDU^{-1}, where U is a unitary matrix and D is a diagonal matrix. Let $\{c_{\rho_i} : i = 1, ...n\}$ be the set of eigenvalues of ρ (and thus the set of eigenvalues of D as well). Since ρ is positive-semidefinite, $c_{\rho_i} \geq 0$ for all $i \leq n$, $i \in \mathbb{N}$. Moreover since $\sum_{i=1}^{n} (c_{\rho_i}) = \text{Trace}(D) = \text{Trace}\left(UU^{-1}D\right) = \text{Trace}\left(UDU^{-1}\right) = \text{Trace}(\rho) = 1$, $c_{\rho_i} \leq 1$ for all $i \leq n$, $i \in \mathbb{N}$ (otherwise an eigenvalue would have to be negative). So we have $c_{\rho_i} \in [0, 1]$ for all $i \leq n$, $i \in \mathbb{N}$.

Note that

$$\text{Trace}\left(\rho^2\right) = \text{Trace}\left(UDU^{-1}UDU^{-1}\right) = \text{Trace}\left(D^2\right) = \sum_{i=1}^{n} c_{\rho_i}^2$$

We'll use induction to show that $\sum_{1=1}^{n} c_{\rho_i}^2 \geq \frac{1}{n}$ for all $n \in \mathbb{N}$.

For the $n = 1$ case, since $\sum_{i=1}^{1} c_{\rho_i} = c_{\rho_1} = 1$, $\sum_{i=1}^{1} c_{\rho_i}^2 = c_{\rho_1}^2 = 1^2 = 1 \geq \frac{1}{1}$, so our hypothesis is true for the $n = 1$ case.

Now if the hypothesis is true for $n = k$, that is, $\sum_{1=1}^{k} c_{\rho_i}^2 \geq \frac{1}{k}$, then

$$\sum_{i=1}^{k+1} c_{\rho_i}^2 = \sum_{i=1}^{k} c_{\rho_i}^2 + c_{\rho_{k+1}}^2 \geq \frac{1}{k} + c_{\rho_{k+1}}^2 > \frac{1}{k+1} + c_{\rho_{k+1}}^2 \geq \frac{1}{k+1}$$

Thus, $\sum_{i=1}^{n} c_{\rho_i}^2 \geq \frac{1}{n}$ for all $n \in \mathbb{N}$.

Now since $c_{\rho_i} \in [0, 1]$, $c_{\rho_i}^2 \in [0, c_{\rho_i}]$, for all $i \leq n$, $i \in \mathbb{N}$, and so $\sum_{i=1}^{n} c_{\rho_i}^2 \leq \sum_{i=1}^{n} c_{\rho_i} = 1$.

Putting these two inequalities together, we get $\frac{1}{n} \leq \sum_{i=1}^{n} c_{\rho_i}^2 \leq 1$, or $\text{Trace}\left(\rho^2\right) \in \left[\frac{1}{n}, 1\right]$ for all $n \in \mathbb{N}$. $\square$

This boundeness makes the trace of ρ^2 a convenient measure to determine how "mixed" a quantum state is, with $\text{Trace}\left[\rho^2\right] = 1$ indicating a "minimally mixed" (i.e. pure) state, and $\text{Trace}\left[\rho^2\right] = \frac{1}{n}$ indicating a "maximally mixed" state, that is, an ensemble of pure states in which each of the pure states has an equal probability of occurring when the system is prepared.

2.1 The partial trace of a density matrix.

The partial trace of a matrix is a generalization of the trace introduced above, and it can be used to determine the entanglement of a system. We will concern ourselves with the four-level quantum system - the simplest system from which entanglement can be extracted. Unlike the trace of a matrix, which yields a number, the partial trace of a matrix yields another operator. In particular, when we take the partial trace of a density matrix, we are free to choose what operator to "trace over." Most interesting for our purposes is tracing over one of our subsystems - either particle A or particle B. If we have a 4×4 density matrix such as the one below:

$$\rho = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the partial trace of ρ over particle B is given by the expression:

$$\mathrm{Trace}_B\,[\rho] = \begin{bmatrix} a_{00} + a_{11} & a_{02} + a_{13} \\ a_{20} + a_{31} & a_{22} + a_{33} \end{bmatrix}$$

This particular partial trace has an illuminating physical meaning: that is, to trace over particle B is to discard all knowledge about the state of particle B, leaving only information about the state of particle A. If the partial trace over B yields a pure state density matrix, then we know that throwing out knowledge about the state of B does not affect our knowledge of the state of particle A. However, if we trace over particle B and find the resulting matrix to be a mixed state, then we know that throwing away infomation about one system automatically creates uncertainty in the state of the other system. In other words, particle A's state cannot be completely described without reference to the state of particle B, and so these particles are entangled. For example, consider the density matrix of the state $|\psi\phi\rangle_{AB} = \frac{1}{2}|0\rangle_A|0\rangle_B - \frac{1}{2}|0\rangle_A|1\rangle_B + \frac{1}{2}|1\rangle_A|0\rangle_B - \frac{1}{2}|1\rangle_A|1\rangle_B$:

$$\rho_0 = |\psi\phi\rangle_{AB}\langle\psi\phi|_{AB} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

ρ_0 has partial trace $\text{Trace}_B[\rho_0] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, we have shown earlier that this represents a pure state, and so ρ_0 is an unentangled state (which is consistent with our earlier construction of $|\psi\phi\rangle_{AB}$ as the product $|\psi\rangle_A \otimes |\phi\rangle_B$, where $|\psi\rangle_A = \frac{1}{\sqrt{2}}|0\rangle_A + \frac{1}{\sqrt{2}}|1\rangle_A$ and $|\phi\rangle_B = \frac{1}{\sqrt{2}}|0\rangle_B - \frac{1}{\sqrt{2}}|1\rangle_B$). Alternately, the state $|\psi\phi\rangle_{AB} = \frac{1}{\sqrt{2}}|0\rangle_A|1\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|0\rangle_B$ has density matrix:

$$\rho_1 = |\psi\phi\rangle_{AB}\langle\psi\phi]_{AB} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and ρ_1 has partial trace $\text{Trace}_B[\rho_1] = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, we have shown earlier that this represents a mixed state, and so ρ_1 is an entangled state (which is also consistent with our previous results).

3 Concurrence

The partial trace of a density matrix is not the only indicator of entanglement. A second method is called concurrence, which is a real number associated with a density matrix. To set up this definition for a given four-level density matrix ρ first consider the matrix $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\, \rho^\dagger\, (\sigma_y \otimes \sigma_y)$, where σ_y is the Pauli spin matrix $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. Then, if $\{\lambda_0,\, \lambda_1,\, \lambda_2\, \lambda_3\}$ is the set of sqaure roots of eigenvalues of the matrix $\rho\tilde{\rho}$, listed in descending order, then the concurrence of ρ, $\mathcal{C}(\rho)$, is defined as

$$\mathcal{C}(\rho) = \max\{0,\, \lambda_0 - \lambda_1 - \lambda_2 - \lambda_3\}$$

Now $\mathcal{C}(\rho) \in [0,\, 1]$, where a concurrence of 0 indicates no entanglement and and a concerrence of 1 indicates maximum entanglement. To show that this definition is consistent with our previous consider again the density matrix

$$\rho_0 = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Now

$$\rho_0\tilde{\rho}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which clearly has 0 its only eigenvalue, and so $\mathcal{C}(\rho_0) = 0$. Thus ρ_0 reperesents an unentangled state, which is the same result we arrived at earlier.

Now consider the density matrix

$$\rho_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Where

$$\rho_0\tilde{\rho_0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $\rho_1\tilde{\rho_1}$ has eigenvalues 1 and 0 (with multiplicity 3), therefore $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, and so $\mathcal{C}(\rho_1) = 1$: ρ_1 is a maximally entangled state (which we have shown earlier using the partial trace). Our ultimate goal is to confirm whether or not an arbitrary four-level quanrum state can be treated as two two-level states with arbitrary entanglement, according to our choice of basis

4 Entanglement is Relative to the Observables

Accoring to Zanardi, Lidar, and Lloyd "A partitioning of a given Hilbert space is induced by the experimentally accessible observables. [...] In this sense entanglement is always relative to a particular set of experimental capabilities." (Source: arXiv:quant-ph/0308043). In the case of Alice and Bob, then, each observer can custom-tailor his or her observables to obtain an arbitrary amount of entanglement. Here, "custom-tailored observables" means choosing an appropriate basis, which can be realized actively (by physically rotating the enitre system) or passively (by redefining the orthonormal basis).

Here, the question of interest is whether or not there is a choice of basis such that any four-level quantum system can appear to be entangled. For pure states, we know that there is such a basis [REFERENCE], but it remains an open question for mixed states. We will prove the existence of such a basis for a particular mixed state.

4.1 Inducing entanglement in a four-level mixed state

4.1.1 A Particular Example

Consider the following mixed state density matrix ρ and unitary matrix U:

$$\rho = \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Straightforward calculations of concurrence result in $\mathcal{C}(\rho) = 0, \mathcal{C}\left(U^{-1}\rho U\right) = 0.6$. So, ρ appears to be unentangled in the standard basis, while ρ appears to be entangled when expressed in the basis consisting of the columns of U.

Interesting, this particular unitary matrix U is the transformation that takes the Bell states - the maximally entangled states - to the standard basis, which are minimally entangled, according to the basis map:

- $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|0\rangle_A|0\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|1\rangle_B \longmapsto |0\rangle_A|0\rangle_B$
- $|\Phi^-\rangle = \frac{1}{\sqrt{2}}|0\rangle_A|0\rangle_B - \frac{1}{\sqrt{2}}|1\rangle_A|1\rangle_B \longmapsto |0\rangle_A|1\rangle_B$
- $|\Psi^+\rangle = \frac{1}{\sqrt{2}}|0\rangle_A|1\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|0\rangle_B \longmapsto |1\rangle_A|0\rangle_B$
- $|\Psi^-\rangle = \frac{1}{\sqrt{2}}|0\rangle_A|1\rangle_B - \frac{1}{\sqrt{2}}|1\rangle_A|0\rangle_B \longmapsto |1\rangle_A|1\rangle_B$

Now if ρ is the density matrix of two particles in the basis shared by observers Alice and Bob, then the unitary transformation U contains the instructions for Alice and Bob to tailor their observables in order for their particles to be entangled. For example, if the product $S_Z^A = \sigma_z \otimes \mathbb{I}$ corresponds to Alice measuring the z-component of spin for her particle and Bob making no measurement on his particle, this particular observable would appear to be unentangled in the original basis; however, when this obserable is expressed in the basis induced by U, $S_Z^P = U^{-1} (\sigma_z \otimes \mathbb{I}) U$ (call it the PQ basis, as opposed to the original AB, "Alice-Bob" basis), then the resulting measurements on the particles would make them appear to be entangled.

4.1.2 General Entanglement Behavior for Classes of Unitary Matrices

We will now use our example density matrix ρ from before to study how its concurrence changes when ρ is transformed by several families of unitary matrices.

First consider the singly-parametrized family of unitary matrices:

$$U_\theta = \begin{bmatrix} \cos(\theta) & 0 & 0 & -\sin(\theta) \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ \sin(\theta) & 0 & 0 & \cos(\theta) \end{bmatrix}$$

Using Mathematica evaluate the concurrence on the interval $\theta \in [0, 2\pi]$, we find that $U_\theta^{-1} \rho U_\theta$ has a maximum concurrence of 0.6 when $\theta = (2n-1)\frac{\pi}{4}$, where $n \in \mathbb{N}$, and has zero concurrence when $\theta = n\frac{\pi}{2}$, $n \in \mathbb{N}$.

Next we have another singly-parametrized family of unitary matrices:

$$U_\theta = \begin{bmatrix} \cos(\theta) & 0 & 0 & \sin(\theta) \\ -\sin(\theta) & 0 & 0 & \cos(\theta) \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \end{bmatrix}$$

(Note that our unitary transformation U in 4.1.1 belongs to this family, as $U = U_{\frac{\pi}{4}}$.) This family exhibits same relationships between θ and $U_\theta^{-1}\rho U_\theta$ as the previous family: $U_\theta^{-1}\rho U_\theta$ has a maximum concurrence of 0.6 when $\theta = (2n-1)\frac{\pi}{4}$, where $n \in \mathbb{N}$, and has zero concurrence when $\theta = n\frac{\pi}{2}$, $n \in \mathbb{N}$.

These two particular examples highlight an observation about the effect of unitary transformations on concurrence: performing a swap opererator S on a unitary transformation will not affect the concurrence of a density matrix; that is, $\mathcal{C}\left((SU)^{-1}\rho(SU)\right) = \mathcal{C}\left(U^{-1}\rho U\right)$. We see that the two families of density matrices shown above are related by a swap operator, as one can be arrived at by permuting the rows and columns of the other.

4.1.3 Further Questions

It remains to be seen whether there exists a mixed state density matrix ρ and a unitary transformation U such that $\mathcal{C}(\rho) = 0$, $\mathcal{C}\left(U^{-1}\rho U\right) = 1$, that is, a choice of basis will transform an unentangled system into one with maximal entanglement.

5 Sources

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