CAPSTONE: COMPARING THE METHODS OF FEJER AND ABEL FOR THE CONVERGENCE OF FOURIER SERIES

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Abstract. Our paper discusses two known methods for the convergence of classical Fourier series, that of the methods of Fejer and Abel. Both exploit the tool of Cesaro convergence and develop the convergence via the techniques of approximating families and the theory of generalized functions. Our paper discusses the differences but also the similarities of these methods, and we develop these from both a mathematical and historical perspective. We close by showing the power of generalized functions across the mathematical spectrum, from differential equations to functional analysis.

In this paper, two known methods for the convergence of classical Fourier series, that of the methods of Fejer and Abel, will be discussed. First, however, we require some background on Fourier series.

Background: Fourier Series

Suppose the following function $f(x)$ of period 2π has the following expansion:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n cos\,(nx) + b_n sin\,(nx)\right). - (*)$$

To solve for the coefficients a_0, a_n, and b_n, we first integrate each side from $-\pi$ to π, as follows:

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{a_0}{2}dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n cos(nx)dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n sin(nx)dx$$

$$= \frac{a_0 x}{2} \mid_{-\pi}^{\pi} + \sum_{n=1}^{\infty}\left[\frac{a_n}{n} sin(nx) \mid_{-\pi}^{\pi} - \frac{b_n}{n} cos(nx) \mid_{-\pi}^{\pi}\right]$$

$$= \pi a_0 + \sum_{n=1}^{\infty} (\sin(\pi n) - \sin(-\pi n)) + \sum_{n=1}^{\infty} (-\cos(\pi n) + \cos(-\pi n))$$

$$= \pi a_0.$$

Then, if we multiply each side of $(*)$ by $\cos(nx)$ and once again integrate from $-\pi$ to π, we obtain the following result for a_n :

$$\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos(nx)dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos^2(nx)dx + b_n \int_{-\pi}^{\pi} \sin(nx)\cos(nx)dx\right)$$

$$= a_n \pi.$$

Similarly, multiplying $(*)$ by $\sin(nx)$ and integrating:

$$\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\sin(nx)dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos(nx)\sin(nx)dx + b_n \int_{-\pi}^{\pi} \sin^2(nx)dx\right)$$

$$= b_n \pi.$$

Thus, we obtain the following Fourier coefficients for the Fourier series of $f(x)$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

We can note that for even functions, where $f(x) = f(-x)$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = 0$, so

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

and for odd functions, where $-f(x) = f(-x)$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0$, and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$, so

$$f(x) \sim \sum_{n=1}^{\infty} b_n \cos(nx).$$

We can similarly write the complex forms of the Fourier series using Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

We know the following trigonometric definitions:

$$\cos(n\theta) = \frac{e^{i(n\theta)} + e^{-i(n\theta)}}{2}$$

$$\sin(n\theta) = \frac{e^{i(n\theta)} - e^{-i(n\theta)}}{2i} = i\left(\frac{-e^{i(n\theta)} + e^{-i(n\theta)}}{2}\right).$$

Applying these definitions to $(*)$, we see the following result:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}e^{inx} + \frac{a_n + ib_n}{2}e^{-inx}\right)$$

Letting $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - ib_n}{2}$, and $c_{-n} = \frac{a_n + ib_n}{2}$, we get:

$$s_m(x) = c_0 + \sum_{n=1}^{m}\left(c_n e^{inx} + c_{-n}e^{-inx}\right) = \sum_{n=-m}^{m} c_n e^{inx}.$$

Therefore,

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Note that we use the notation $f(x) \sim \frac{a_0}{2} + \sum\left(a_n\cos(nx) + b_n\sin(nx)\right)$ if we do not know whether or not the Fourier series converges to the function $f(x)$. We write $f(x) = \frac{a_0}{2} + \sum\left(a_n\cos(nx) + b_n\sin(nx)\right)$ only after we have proven that the series converges to $f(x)$.

Consider an infinite series of functions

$$\sum_{k=1}^{\infty} f_k(x),$$

where

$$s_n(x) = \sum_{k=1}^{\infty} f_k(x),\ n \in \mathbb{N}^+.$$

We say that the series is convergent for a value x if its partial sums have a finite limit

$$s(x) = \lim_{n\to\infty} s_n(x).$$

If a $s(x)$ exists for all x in an interval $[a, b]$, then the series is said to be convergent on that interval.

Further, the series is uniformly convergent on an interval $[a, b]$ if for any $\varepsilon > 0$, $\exists N$ such that $\mid s(x) - s_n(x) \mid \leq \varepsilon \ \forall n \geq N \ \forall x \in [a, b]$.

We can test for uniform convergence using the Weirstrass M Test. If the series of positive numbers $M_1 + M_2 + ... + M_k + ...$ converges and if for any x in the interval $[a, b]$, $\mid f_j(x) \mid \leq M_j$ for all $j \geq k$, then the series converges uniformly.

We can now consider the following theorems regarding convergence of a Fourier series:

Theorem 1. If a function $f(x)$ of period 2π can be expanded in a trigonometric series which converges uniformly on the whole real axis, then this is the Fourier series of $f(x)$. We might also note that because $f(x)$ is 2π periodic, uniform convergence on any closed interval of length 2π is equivalent to uniform convergence on the whole real axis.

Theorem 2. We have the additional conclusion that if an absolutely integrable function $f(x)$ (again, $f(x)$ is 2π periodic) can be expanded in a trigonometric series which converges to $f(x)$ everywhere except possibly at a finite number of points, then this is the Fourier series of $f(x)$.

Summation of trigonometric fourier series

Suppose we are given a trigonometric Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

of some function $f(x)$. How can we determine that function $f(x)$? Does our ability to find the function change depending on the convergence of the series? We can see that if the trigonometric series above does in fact converge to $f(x)$, then the function $f(x)$ is the limit of the partial sums of the series. However, how do we find $f(x)$ when we do not know whether or not the series converges, or we know that the series diverges? In these cases, we either know that the limit of partial sums does not exist, or we are unsure of whether or not it does. The operation that can be used to determine $f(x)$ solely from its Fourier series, without knowledge of its convergence or divergence, is summation.

Origin of the Theory of Summability

The origin of the theory of summability of series lies in a letter written by Leibniz in 1713 in which he considered the series $1 + 1 - 1 + ...$, and Guido Grandi's assignment of $\frac{1}{2}$ as the value of this series. Although aware that his argument was a significant departure from the conventional mathematical procedure, the reasoning he provided was essentially the following: the sum of an even number of terms in the series is always equal to zero, whereas the sum of an odd number of terms in the series is always equal to one. However, we must consider an infinite number of terms, which of course, can neither be defined as an even nor an odd number of terms. Thus, neither zero nor one can be assigned as the sum of an infinite number of terms, and thus, an intermediate value is appropriate. Because the sums zero and one are equally likely as the number of terms approaches infinity, the sum of an infinite number of terms is simply the mean of zero and one—$\frac{1}{2}$.

In 1771, Daniel Bernoulli introduced the first extension of Leibniz's process using mean values. Bernoulli defined the period of a recurrent series to be the set of terms which recur and have sum zero. For example, in the series $1 + 1 - 1 + ...$, the period is $1 - 1$. Defining n as the number of terms in the period and s_{k-1} as the sum of the first k terms $u_0, u_1, ..., u_k$, Bernoulli created the following equation for the sum of the series:

$$\frac{s_0 + s_1 + ... + s_{n-1}}{n}.$$

In the late 1700's and early 1800's, Lagrange and Raabe began in the direction of a rigorous treatment of summable divergent series by providing proofs that $\lim[u_0 + u_1 x + u_2 x^2 + ...]$ exists and is equal to the sum of the series, as given above, for recurrent series. Although Leibniz and Bernoulli certainly considered the same idea, it was Leibniz and Raabe who first looked past specific examples and towards a general theory.

Later, in 1880, Frobenius published an extension of Lagrange and Raabe's theorem, in which he generalized recurrent series where the expression repeats itself at regular intervals to consider series where the sum tends to a definite limit as n approaches infinity. Soon after, Hölder published a paper introducing the method of summation by successive means and showing that a series can be summed by using convergence factors. A well-known paper by Cesaro then introduced the idea of summability by weighted means.

The result found in Poisson's integral is simply a method of summing the Fourier series of an arbitrary function $f(x)$ using convergence factors. Fejer soon began investigating the possibility of summing Fourier series by mean value methods, where his result regarding summation of Fourier series by arithmetic means sparked extensive further study in the field of summation of series by mean value methods.

Two Known Methods

We now look to two known methods for the convergence of classical Fourier series, that of the methods of Fejer and Abel. Both exploit the tool of Cesaro convergence and develop the convergence via the techniques of approximating families and the theory of generalized functions. Let us note that the two methods of convergence follow the same four steps, but do so in difference ways, as we will see. For both Fejer's and Abel's methods of convergence, we notice these basic steps:

1. The summation is written as a convolution integral by exchanging sums and integrals.

2. The kernel of the integral equation is shown to form an approximate identity family.

3. It is shown that if we push the index of the approximate identity family, it acts more and more like the dirac delta.

4. Convolution of the function $f(t)$ with the dirac delta produces the original function.

We first explore the method of Abel.

Consider the series $u_0 + u_1 + ... + u_n + ...$ (A) and the series $u_0 + u_1 r + u_2 r^2 + ... + u_n r^n + ...$(B). Assuming that for $0 < r < 1$, the second series (B) converges and $\lim_{r\to 1} \sigma(x) = \sigma$ exists, where $\sigma(r)$ is the sum of (B). Thus, the first series is summable by Abel's method to the value σ.

It is important to note that if the series (A) converges and its sum is equal to σ, then (A) is summable by Abel's method to the same number σ. In other words, if a series converges, Abel's method of summation gives the same value as the sum in the usual sense of summing a series. We can prove this proposition using the fact that if the series (A) is convergent, then the series (B) converges for $0 \le r \le 1$, and its sum $\sigma(r)$ is continuous on the interval $[0, 1]$. Thus, if the series (A) converges, $\lim_{r\to 1} \sigma(r) = \sigma(1) = \sigma$.

We can now apply Abel's method to the summation of Fourier series. Suppose we have an absolutely integrable function $f(x)$ such that

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

Since a_n and b_n both approach zero as $n \to \infty$, the series

$$f(x, r) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

converges for $0 \leq r < 1$. Therefore, $\mid a_n \mid \leq M$ and $\mid b_n \mid \leq M$ for $n = 1, 2, ...$ and M a constant.

Since $\mid \cos(nx) \mid \leq 1$ and $\mid \sin(nx) \mid \leq 1$, $\mid r^n \left(a_n \cos(nx) + b_n \sin(nx) \right) \mid \leq 2Mr^n$.

If $\lim_{r \to 1} f(x,r)$ exists, then the series $f(x,r)$ is summable by Abel's method.

If we use

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(nt)dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(nt)dt$$

to rewrite $f(x,r)$ as the convolution integral

$$f(x,r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi} \sum_{n=1}^{\infty} r^n \int_{-\pi}^{\pi} f(t)\cos n(t-x)dt,$$

we can see that for a fixed $r < 1$, the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(t-x)$$

converges uniformly using a comparison with the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n,$$

which we know to converge.

Then, we can rewrite our above convolution integral

$$f(x,r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi} \sum_{n=1}^{\infty} r^n \int_{-\pi}^{\pi} f(t)\cos n(t-x)dt$$

as the following:

$$f(x,r) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(t-x) \right] dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1-r^2}{1-2r\cos(t-x)+r^2} dt,$$

with $0 \leq r < 1$.

Note that we use Poisson's kernel here:

$$\frac{1-r^2}{1-2r\cos\varphi+r^2} \geq 0.$$

Now, we have written $f(x,r)$ as an integral using Poisson's kernel. Since Poisson's kernel forms an approximate identity family with index r, as we let $r \to 1$, the approximate identity family acts more and more like the dirac delta. Convolving this with $f(x)$ then yields the original function $f(x)$.

We can now explore Fejer's methods of convergence. When looking at the question of whether or not a continuous function $f(x)$ from $\mathbb{T}$ to $\mathbb{C}$ can be determined from its Fourier coefficients, Fejer showed at the age of 19 that the answer is yes. Cesaro, and later, Fejer, both explored the idea of considering averages in order to "improve" the behavior of a badly behaved function. He studied the behaviors of s_0, $(s_0+s_1)/2$, $(s_0+s_1+s_2)/3$, ... to make conclusions about the sequence s_0, s_1, s_2, ...

Theorem. (Fejer): Let $f:\mathbb{T}\to\mathbb{C}$ be Riemann integrable. Then if f is continuous at t, then

$$\sigma_n(f,t) = \sum_{k=-n}^{n} \frac{(n+1)-\mid k\mid}{(n+1)}\hat{f}(k)e^{ikt}$$

converges point-wise to $f(t)$ as $n\to\infty$. Additionally, if $f:\mathbb{T}\to\mathbb{C}$ is continuous, then $\sigma_n(f,t)$ convergest uniformly to $f(t)$ as $n\to\infty$.

We can now rewrite $\sigma_n(f,t)$ to a convolution integral with kernel $K_n(t)$ and then show that $K_n(t)$ (Fejer's kernel) is an approximate identity family with index n.

Note that $K_n(t)$ is the Fejer kernel, and

$$K_n(t) = \sum_{k=-n}^{n} \frac{(n+1)-\mid k\mid}{(n+1)}e^{ikt}.$$

By the definition of $\hat{f}$,

$$\sigma_n(f,t) = \sum_{k=-n}^{n} \frac{(n+1)-\mid k\mid}{(n+1)}\left[\hat{f}(k)\right]e^{ikt} = \sum_{k=-n}^{n} \frac{(n+1)-\mid k\mid}{(n+1)}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx\right]e^{ikt}.$$

By linearity,

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sum_{k=-n}^{n} \frac{(n+1)-\mid k\mid}{(n+1)}e^{-ik(t-x)}dx$$

By the above definition of the Fejer kernel, $K_n(t)$,

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)K_n(t-x).$$

We then use a change of variables and the 2π periodicity of f to say

$$= \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} f(t-y)K_n(y)dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-y)K_n(y)dy.$$

Thus, we have written the sum as a convolution integral. Note the following properties of the Fejer kernel:

1. $K_n(t) \geq 0 \ \forall t$
2. $K_n(t)$ converges uniformly to 0 outside of $[-\delta, \delta]$ $\forall \delta$, $0 < \delta < \frac{\pi}{2}$, as $n \to \infty$
3. $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1 \ \forall n$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-n}^{n} \frac{(n+1)-\mid k \mid}{(n+1)} e^{ikt}dt$$

$$= \sum_{k=-n}^{n} \frac{(n+1)-\mid k \mid}{(n+1)} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}dt \right) = \sum_{k=-n}^{n} \frac{(n+1)-\mid k \mid}{(n+1)} (\delta_{0k})$$

$$= \frac{n+1}{n+1} = 1.$$

Thus, the Fejer kernel forms an approximate identity family with parameter n.

We can see that as we push the parameter n, the approximate identity family behaves more and more like the dirac delta (see figure A), which, when convolved with the function $f(t)$, produces the original funtion $f(t)$.

Conclusion

Thus, both Fejer's and Abel's methods of convergence write the summation as a convolution integral by exchanging sums and integrals, show that the kernel of the integral equation is shown to form an approximate identity family, show that if we push the index of the approximate identity family, it acts more and more like the dirac delta, and convolves the function $f(t)$ with the dirac delta to produce the original function. Now that we have seen that both Fejer and Abel convergence follow the same four basic steps, where do these methods differ?

These two methods for summation differ in the tricks employed to follow the four-step recipe above. Primarily, Fejer employs Cesaro summation on the unit circle to form the approximate identity family. Alternatively, Abel makes use of the Poisson kernel and comparisons of geometric series inside the unit circle.

We have seen how two known methods for the convergence of classical Fourier series approach the same problem in ways that follow the same overall recipe, but use different tools to do so. Both exploit the tool of Cesaro convergence and develop the convergence via the techniques of approximating families and the theory of generalized functions, and have shown us the power of generalized functions via the convolution with the dirac delta as an “identity function”. It is interesting to consider the remarkable similarities in these two methods, as Fejer and Abel were essentially following two parallel paths to convergence of Fourier series.

Works Cited

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